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HIGHER ORDER SCHWARZIANS FOR GEODESIC FLOWS, MOMENT SEQUENCES, AND THE RADIUS OF ADAPTED COMPLEXIFICATIONS

RAUL M. AGUILAR

Science and Mathematics Department
Massachusetts Maritime Academy
Buzzards Bay, MA, 02532
e-mail: raguilar@maritime.edu
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1. Abstract

In the first part of the paper, comprising section 1 through 6, we introduce a sequence of functions in the tangent bundle TM of any smooth two-dimensional manifold M with smooth Riemannian metric g that correspond to the higher order Schwarzians of the linearized geodesic flow. With these functions and a classical theorem of Loewner on analytic continuation we are able to characterize the existence of the adapted complex structure induced by g on the set T^RM of vectors in TM of length up to R, equivalently for M compact, to the existence of a Grauert tube of radius R in terms of infinite Hankel matrices involving these Schwarzian functions. The basic characterization so obtained can be expressed as a sequence of differential inequalities of increasing order polynomial in the covariant derivatives of the Gauss curvature on M and in πR^{-1} that should be regarded as the higher order versions of a curvature inequality by L. Lempert and R. Szőke.

The second part of the paper, sections 7 through 11, includes a discussion of the rank of the infinite Hankel matrix of the Schwarzians from part 1 and of new Schwarzians defined now for purely imaginary radius, as well as some computations and examples. A characterization of the existence of the adapted structure on T^RM in terms of moment sequences with parameters R and v in TM is also noted.

Contents

2. Introduction and description of main results

Let M be a two-dimensional real analytic complete Riemannian manifold with metric g and Gauss curvature σ, and let its tangent bundle be denoted by $\pi\colon TM \to M$.

L. Lempert and R. Szőke [17] defined for any real analytic metric g the *adapted complex structure*, which always exists in some neighborhood of the zero section $M \subset TM$. An equivalent construction is defined in the cotangent bundle by V. Guillemin and M. Stenzel [11] (Please see beginning of Section 6 for definitions).

A prototype for these structures is found in the study of the geometry of certain foliations in complexifications of symmetric spaces of rank one by G. Patrizio and P.M. Wong [19], which in turn was motivated by the work on the complex homogeneous Monge-Ampére equation by D. Burns [7] and that of W. Stoll [22].

There is current interest in the adapted complex structure, and from various points of view [14] [23]

We have concentrated our attention in how the geometry of (M, g) determines the set where the adapted complex structure can be defined. One aspect of this is the determination of the possible values of $R > 0$ for which

$$T^R M \;=\; \{\mathrm{v} \in TM, \|\mathrm{v}\| = \sqrt{g(\mathrm{v},\mathrm{v})} \;<\; R\} \tag{2.1}$$

admits the adapted complex structure, equivalently, for M compact, the possible radii R the Grauert tube asssociated to (M, g) can have.

There are two inequalities that serve as our motivation and starting point.

The first one is due to Lempert and Szőke [17] who showed that if the adapted complex structure is defined on $T^R M$, for some $R > 0$ finite or infinite, the inequality

$$\boxed{\sigma \;\geq\; -\frac{1}{4}\frac{\pi^2}{R^2}} \tag{2.2}$$

must hold. The second one applies to the case $R = \infty$ and is due to Szőke [**?**] who proved that in addition to (2.2), if the adapted structure is defined on all TM, the inequality

$$\Delta\sigma \;\geq\; -16\,\sigma^2\,, \tag{2.3}$$

with Δ the Laplacian operator for the metric g on M, must hold as well.

Clearly, the curvature condition alone is far from sufficient to guarantee existence of the structure on $T^R M$. This is illustrated for $R = \infty$ by the case of a torus [17], by Szőke result on surfaces of revolution [24], and as shown by the author, by the triaxial ellipsoid [3] and more generally Liouville metrics on the two-sphere [2].

So, we would like to understand if there are properties of the curvature that when met would guarantee the existence of the adapted structure. This question of course applies to M of any dimension, but as a first step we limit ourselves to dimension of $M = 2$; so "curvature" means Gauss curvature.

The two necessary conditions above are a consequence of the non-negativity of all the derivatives of odd order of functions in the Pick class, also known as Nevanlinna or Herglotz functions, that is, functions analytic and with positive imaginary part on the upper half-plane,

$$H^+ \;:=\; \{z \in \mathbb{C}\,|\,0 \;<\; \Im z\}, \tag{2.4}$$

which are defined across some interval of the boundary of H^+. Higher order necessary conditions are of course obtainable, as the computations of Lempert and Szőke make clear.

In this paper we propose a framework to view all those higher order conditions, and provide *necessary and sufficient* conditions for the adapted complex structure to be defined on $T^R M$ that depend on the Gauss curvature σ of M. This is possible because our set up is based on properties of derivatives which give a characterization of functions in the Pick class according to Loewner's Theorem [6], [9].

We give a basic set of conditions in the form of an infinite sequence of differential inequalities of increasingly higher order in σ, viewed as function in TM.

Some of our results will apply to sectional curvature in higher dimensions for some situations, giving necessary conditions. However obtaining necessary and sufficient conditions requires the curvature operator.

In (2.16) we display our basic inequalities in terms of certain functions in TM, the "higher order Schwarzians in TM", that we introduce in Section 5.2. These inequalities are preceded by those in (2.19) that we derive in Section 3 in terms of the higher order Schwarzians on the real line $\mathbb{R}$.

For convenience we introduce the following.

Definition 2.1. *Let $R \in (0, \infty]$ be fixed. We denote by $\mathfrak{L}_R$ the class of smooth functions $\vartheta \colon \mathbb{R} \to \mathbb{R}$ for which a quotient, and hence* all *quotients (see comment at the beginning of the Proof of Theorem 1) of independent solutions of*

$$y''(x) + \vartheta(x)\, y(x) \,=\, 0 \tag{2.5}$$

have meromorphic extensions to the R-strip

$$\{z \in \mathbb{C}\,, |\Im z| \,<\, R\}$$

that are holomorphic and with non-vanishing imaginary part for $0 \,<\, \Im z \,<\, R$.

Thus, we begin by seeking conditions on ϑ to be in $\mathfrak{L}_R$. It is well-known, and can be checked by a computation, that ϑ is related to the quotient $h \,=\, y_2/y_1$ of any pair of independent solutions of (2.5), y_1 and y_2, by the equality valid off the zeros of y_1,

$$2\,\vartheta \,=\, \frac{h'''}{h'} - \frac{3}{2}\left(\frac{h''}{h'}\right)^2 , \tag{2.6}$$

the right-hand side of which is the *(classical) Schwarzian* of h. For high order conditions on ϑ to be in $\mathfrak{L}_R$ we are led to consider the *higher order Schwawrzians*

$$\mathcal{S}_n^\vartheta \colon \mathbb{R} \to \mathbb{R}, \tag{2.7}$$

for $n \,=\, 1,\, 2,\, \cdots$, defined by the generating function (see also (3.4))

$$\frac{h(x+t) - h(x)}{h'(x) + \frac{1}{2}\,\frac{h''(x)}{h'(x)}\,(h(x+t) - h(x))} \,=\, \sum_{n\,=\,1}^{\infty} \mathcal{S}_n^\vartheta(x)\, \frac{t^n}{n!}\,. \tag{2.8}$$

Remark 2.1. *By the Faà di Bruno formula,*

$$\frac{\mathcal{S}_n^\vartheta(x)}{n!} = \sum_{\substack{k\,=\,1 \\ k_1 + \cdots + k_n = k \\ k_1 + 2k_2 \cdots + nk_n = n}}^{n} (-1)^{k+1} \frac{k!}{k_1! \cdots k_n!} \left(\frac{h^{(1)}(x)}{1!}\right)^{k_1 - 2k + 1} \left(\frac{h^{(2)}(x)}{2!}\right)^{k_2 + k - 1} \cdots \left(\frac{h^{(n)}(x)}{n!}\right)^{k_n} .$$

always provided that $h'(x) \neq 0$, otherwise use $1/h(x)$, in light of Remark 3.2.

Remark 2.2. *We get that $\mathcal{S}_0^\vartheta \,=\, 0$, $\mathcal{S}_1^\vartheta \,=\, 1$, $S_2^\vartheta \,\equiv\, 0$, while $\mathcal{S}_3^\vartheta$ equals the right-hand side of (2.6). Moreover (see Section* **??***)*

$$\mathcal{S}_4^\vartheta = \frac{h''''}{h'} - 4\frac{h''\, h'''}{(h')^2} + 3\left(\frac{h''}{h'}\right)^3 \,=\, 2\,\vartheta'\,. \tag{2.9}$$

These functions are treated by H. Tamanoi [25]*, and more recently by S. Kim and T. Sugawa* [12]*. Note that our indexing for the Schwarzians is shifted by one unit from theirs. The following recursion, hinted at in (2.9), gives an alternative definition (Proposition 2.1* [12]*),*

$$\mathcal{S}_{n+1}^\vartheta(x) \,=\, \left(\mathcal{S}_n^\vartheta(x)\right)' \,+\, \vartheta(x) \sum_{k=1}^{n-1} \binom{n}{k}\, \mathcal{S}_k^\vartheta(x)\, \mathcal{S}_{n-k}^\vartheta(x)\,. \tag{2.10}$$

Remark 2.3. *In Proposition 3.3, we provide another very useful expression for these functions that can be interpreted as their "normal coordinate" definition; in fact that is the formula most used throughout this paper.*

By definition, for $R = \infty$ the function ϑ is in $\mathfrak{L}_R$ if and only if any quotient h of two independent solutions (2.5) is a Pick function defined also across intervals in $\mathbb{R} = \partial H^+$. These functions are classically characterized by the non-negativity of an infinite quadratic form defined by their derivatives arranged in a Hankel matrix of infinite order. This leads to inequalities that are determinantal of increasing order in the Schwarzians, and thus, to canonical higher order conditions for ϑ.

We treat the case $R < \infty$ simultaneously with the case $R = \infty$ by the introduction of higher order "g_R-Schwarzians"

$$\mathcal{S}_n^{\vartheta,\,g_R}\colon \mathbb{R} \to \mathbb{R}$$

given by Definition 3.2. These are linear combinations of the Schwarzians, as displayed in (3.14), with coefficients polynomial in R that depend on a choice of a Pick function g_R selected so that in the limit $R = \infty$ the g_R-Schwarzians become the higher order Schwarzians.

Remark 2.4. *To illustrate, with* $g_R(z) = -\dfrac{R}{\pi}\ln\Big(1 - \dfrac{\pi\, z}{R}\Big)$*, we now list (see Section* **??***),*

$$\mathcal{S}_1^{\vartheta,\,g_R} = 1\,, \quad \mathcal{S}_2^{\vartheta,\,g_R} = \frac{\pi}{R}\,, \quad \mathcal{S}_3^{\vartheta,\,g_R} = \mathcal{S}_3^{\vartheta} + \frac{2\,\pi^2}{R^2} = 2\,\vartheta + \frac{2\,\pi^2}{R^2}\,. \tag{2.11}$$

The main results are the following.

Theorem 1 shows that, for R finite or infinite, ϑ is in $\mathfrak{L}_R$ if and only if for all integer $n \geq 1$ and all $x \in \mathbb{R}$,

$$\mathcal{D}_n^{\vartheta,\,g_R}(x) := \det\left[\frac{\mathcal{S}_{i+j-1}^{\vartheta,\,g_R}(x)}{(i+j-1)!}\right]_{i,\,j=1}^{n} \geq 0\,. \tag{2.12}$$

Furthermore, assuming real analyticity of ϑ, it is a sufficient condition that the inequalities hold at at least one point.

Since the functions $\mathcal{S}_n^{\vartheta}$ are expressible as polynomials in ϑ and its derivatives,

$$\mathcal{S}_n^{\vartheta} = \pi_n(\,\vartheta,\,\vartheta',\,\ldots,\,\vartheta^{(n-3)}), \tag{2.13}$$

where π_n are certain polynomials, essentially the ones given by Tamanoi [25] (See Proposition 5.2 and Remark 2.3), then the inequalities (2.12) yield canonical conditions on ϑ.

It follows that for every integer $n \geq 2$ the inequality $\mathcal{D}_n^{\vartheta,\,g_R} \geq 0$ is polynomial in

$$\Big\{\vartheta,\,\vartheta',\,\vartheta'',\,\cdots,\,\cdots,\,\vartheta^{(2n-4)},\,\frac{\pi}{R}\Big\}\,,$$

of degree $n(n-1)$ in $\frac{\pi}{R}$, isobaric with weight $\mathrm{w} = n(n-1)$ in ϑ and its derivatives (See Definition 5.1) and with rational constant coefficients.

The construction in Theorem 1 is readily applied to quotients of Jacobi fields along the geodesics of a two-dimensional manifold M with Riemannian metric g. One obtains necessary and sufficient conditions for the existence of the adapted complex structure on $T^R M$, for R finite or infinite. Such characterizing conditions are expressible as inequalities in covariant derivatives of the Gauss curvature σ on M.

In more detail, for a given g_R as above, define a sequence of functions on the tangent bundle

$$\mathrm{S}_n^{\sigma,\,g_R}\colon\ TM \to \mathbb{R}, \tag{2.14}$$

so that restricted to the unit tangent bundle $UM \subset TM$ correspond to the higher order g_R-Schwarzians for the second order differential equation associated by linearization to the geodesic flow geodesic flow

$$\phi\colon UM \times \mathbb{R} \to UM\,; \tag{2.15}$$

here $\phi_t \mathrm{v} = \dot\gamma(t)$ if γ is the geodesic with $\dot\gamma(0) = \mathrm{v}$. Thus, we have for all integer $n > 1$, all $\mathrm{v} \in UM$ and all $t \in \mathbb{R}$,

$$\mathrm{S}_n^{\sigma,\,g_R}(\phi_t \mathrm{v}) = \mathcal{S}_n^{\vartheta,\,g_R}(t)\,, \tag{2.16}$$

where

$$\vartheta(t) = \sigma(\phi_t \mathrm{v})\,. \tag{2.17}$$

Putting for $\mathrm{v} \in TM$,

$$\|\mathrm{v}\|^2 = g(\mathrm{v}, \mathrm{v}),$$

the correspondence (2.16) formally amounts to replacing in (2.13) the function $\vartheta = \vartheta(t)$ by the product of $\|\mathrm{v}\|^2$ times the Gauss curvature σ viewed as a function in TM, and the derivatives of ϑ with respect to t in (2.13) by derivatives of σ along the geodesic flow.

We introduce these functions in Definition 5.2 and Definition 5.3 where we use, equivalently, covariant derivatives of σ. Here we note for clarification that in particular for $R = \infty$,

$$\mathrm{S}_n^\sigma = \pi_n(\|\mathrm{v}\|^2 \sigma, \|\mathrm{v}\|^2 \nabla_{\mathrm{v}} \vartheta, \ldots, \|\mathrm{v}\|^2 \|\nabla_{\mathrm{v}}^{n-3} \sigma), \tag{2.18}$$

and that the corresponding functions for finite R are linear combinations of the functions in (2.18) with coefficients polynomial in $R\,\|\mathrm{v}\|^{-1}$.

Theorem 2 characterizes the existence of the adapted complex structure on $T^R M$ where $R > 0$, is finite or infinite, by a basic sequence of inequalities

$$\mathrm{D}_n^{\sigma,\, g_R}(\mathrm{v}) := \det \left[\frac{\mathrm{S}_{i+j-1}^{\sigma,\, g_R}(\mathrm{v})}{(i+j-1)!} \right]_{i,\, j=1}^{n} \geq 0\,, \tag{2.19}$$

for all integer $n \geq 1$ and for all v on a certain set $\mathcal{Z} \subset TM$ which may be taken to be the unit tangent bundle UM,[1] Such are inequalities polynomial in the covariant derivatives of σ illustrated by (9.15).

Remark 2.5. *From this basic set, by integration along each fiber of the unit tangent bundle UM, one may derive necessary conditions on M which include (2.2) and, in second order in σ, as shown in Proposition 9.2,*

$$32\,\sigma^3 + 6\,\Delta(\sigma^2) + \left(3\,\Delta\sigma + 48\,\sigma^2\right)\frac{\pi^2}{R^2} + 18\,\sigma\,\frac{\pi^4}{R^4} + 2\,\frac{\pi^6}{R^6} \geq 27\,\|\,\mathrm{Grad}\,\sigma\|^2\,. \tag{2.20}$$

Note that inequality (2.20) results from a determinantal condition in the quadratic form referred to above, and in the limit $R = \infty$ reduces to an inequality stronger than (2.3) which comes from a diagonal condition.

Remark 2.6. *One more integration, now of (2.20) on M assumed orientable and closed, yields, with $\int_M \mathrm{dA} = \mathrm{Area}(M)$ the total area of M,*

$$\begin{aligned} 32 \int_M \sigma^3\,\mathrm{dA} + 48\,\left(\tfrac{\pi}{R}\right)^2 \int_M \sigma^2\,\mathrm{dA} + 18\,\left(\tfrac{\pi}{R}\right)^4 \int_M \sigma\,\mathrm{dA} + 2\,\left(\tfrac{\pi}{R}\right)^6 \mathrm{Area}(M) \\ \geq 27 \int_M \|\,\mathrm{Grad}\,\sigma\|^2\,\mathrm{dA}\,. \end{aligned} \tag{2.21}$$

In Theorem 2 the role of the Schwarzians is to provide an intrinsic description, via determinantal curvature inequalities, of the higher order properties of Jacobi fields necessary for the existence of the adapted complex structure. However, the infinite Hankel matrix with the entries involving the g_R-Schwarzians can be used to characterize other properties of quotients of Jacobi fields along a geodesic. We include some results along this direction in Section 7 as well as in Section 8 where we extend the parameter R in the g_R-Schwarzians to purely imaginary values, $R = \sqrt{-1}\lambda$.

In Section 10 we briefly point out some equivalent ways in which properties of the Schwarzians characterize the existence of the adapted structure. This is derived from the classical theory of moments applied at each $\mathrm{v} \in TM$, which for example implies that the existence of a positive Borel measure so that for all integer $n \geq 1$

$$\boxed{\mathrm{S}_n^{\sigma,\, R}(\mathrm{v}) = n! \int_{-\infty}^{\infty} t^{n-1} \mathrm{d}\mu_{\mathrm{v}}^{\sigma,\, R}(t)}$$

is equivalent to the adapted complex structure being defined on $T^R M$. Here the measures depend on v, but uniform estimates of the size of their support, all of which are actually bounded, are easily obtainable under certain assumptions on the Gauss curvature in M. For completeness we include these in Proposition 10.1.

[1]Since our definitions render $\mathrm{D}_n^{\sigma,\, g_R}$ homogeneous along the fiber of TM of degree $n(n-1)$, we may take any set that project (along the fibers of TM) to UM, or even a smaller is there is symmetry

We include many illustrative computations in section 9 such as the explicit calculation of the Schwarzians and related determinants for constant curvature. A list items included in this section is displayed in the table of contents.

3. Higher Schwarzians in $\mathbb{R}$ and related functions

Let $\vartheta\colon \mathbb{R} \to \mathbb{R}$ be a smooth function and f_1 and f_2 any pair of independent solutions of

$$f''(t) + \vartheta(t) f(t) = 0\,. \tag{3.1}$$

Since $f_1\, f_2' - f_1'\, f_2 \neq 0$

$$\mathbf{F}(t) \;=\; \begin{bmatrix} f_1(t) & f_1'(t) \\ f_2(t) & f_2'(t) \end{bmatrix} \tag{3.2}$$

is invertible for all t; thus, for all (s,t) in $\mathbb{R}^2$, we define the matrix

$$\mathbf{F}(x,\, t) \;:=\; \mathbf{F}^{-1}(x)\, \mathbf{F}(t)\,.$$

For fixed x, the components of the first column of $\mathbf{F}(x,\, t)$ viewed as functions of t, form a pair of independent solutions of (3.1). Since $\mathbf{F}(x,\, x) \;=\; \mathbf{I}$, the 2×2 identity matrix, the function

$$V(x,\, t) \;=\; \frac{\mathbf{F}(x,\, t)_{2,1}}{\mathbf{F}(x,\, t)_{1,1}}, \tag{3.3}$$

is defined for all values of t in a neighborhood of x (See also Remark 3.1).

Definition 3.1. *The* higher order Schwarzian of order n corresponding to ϑ *is the function* $\mathcal{S}_n^{\vartheta}\colon \mathbb{R} \to \mathbb{R}$ *given by*

$$\mathcal{S}_n^{\vartheta}(x) \;:=\; \frac{\partial^n V(x, x+t)}{\partial\, t^n}\, |_{\,t=0}\,. \tag{3.4}$$

Remark 3.1. *Let $x \in (a,b) \subset \mathbb{R}$ and let $h \;=\; f_2/f_1$ where f_1 and f_2 independent solutions of (3.1) with $f_1 \neq 0$ in (a,b). Then, it follows, using (3.3), $h \;=\; \frac{f_2}{f_1}$, $h' \;=\; \frac{f_2'}{f_1} - \frac{f_2\, f_1'}{f_1^2} \;=\; \frac{1}{f_1^2}$ and $h'' \;=\; -2\, \frac{f_1'}{f_1^3}$ that*

$$V(x,\, x+t) \;=\; \frac{h(x+t) - h(x)}{h'(x) + \frac{1}{2}\, \frac{h''(x)}{h'(x)}\, (h(x+t) - h(x))}\,. \tag{3.5}$$

Thus the functions in Definition 3.1 are the same as those in (2.8).

Remark 3.2. *It is well-known that the values in (3.4) do not depend on the pair of solutions chosen in (3.2), since given another pair of independent solutions $\tilde f_1$ and $\tilde f_2$ there are a,b,c,d real constants with $a\,d \;\neq\; b\,c$ so that $\tilde{\mathbf{F}}(t) \;:=\; \begin{bmatrix} \tilde f_1(t) & \tilde f_1'(t) \\ \tilde f_2(t) & \tilde f_2'(t) \end{bmatrix} \;=\; \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} f_1(t) & f_1'(t) \\ f_2(t) & f_2'(t) \end{bmatrix}$, and hence $\tilde{\mathbf{F}}(s,\, t) \;:=\; \tilde{\mathbf{F}}^{-1}(x)\, \tilde{\mathbf{F}}(t) \;=\; \mathbf{F}^{-1}(x)\, \mathbf{F}(t) \;=\; \mathbf{F}(x,\, t)$.*

Definition 3.2 (The g_R-Schwarzians)**.** *For $R \;\in\; (0,\, \infty]$ we denote by g_R a Pick function*

$$g_R : (-a,\, a) \cup (\mathbb{C} \setminus \mathbb{R}) \;\to\; (-b, b) \cup \{z \;\in\; \mathbb{C} \,|\, 0 \;<\; |\Im z| \;<\; R\} \tag{3.6}$$

for some $a,b \in \mathbb{R}$, $a,b > 0$, so that $g_R(0) = 0$, and

$$\lim_{R \mapsto \infty}\, g_R(z) \;=\; z \tag{3.7}$$

for every z in its domain. Then, for any such g_R, we define higher order "g_R-Schwarzians" by

$$\boxed{\mathcal{S}_n^{\vartheta,\, g_R}(x) \;=\; \frac{\partial^n}{\partial\, t^n} V\left(x,\; x \;+\; g_R(t)\right)\, |_{\,t=0}}\,. \tag{3.8}$$

Remark 3.3. *A preferred example is $g_R(z) \;=\; -\dfrac{R}{\pi}\, \ln\Big(1 - \dfrac{\pi\, z}{R}\Big)$. We will indicate when such a specific choice is made.*

We present preliminary formulas to be used in Theorem 1.

Definition 3.3. *For a given $x \in \mathbb{R}$* the pair of fundamental solutions associated to the point *x is the pair of solutions $f_{x,1} = f_{x,1}(t)$ and $f_{x,2} = f_{x,2}(t)$ of (3.1) determined by*

$$f_{x,1}(x) = f'_{x,2}(x) = 1 , \quad f_{x,2}(x) = f'_{x,1}(x) = 0 . \tag{3.9}$$

Proposition 3.1. *Given $x \in \mathbb{R}$ putting $h_x(t) := f_{x,2}(t)/f_{x,1}(t)$ then for integer $n \geq 1$, for any g_R as in Definition 3.1 and with the derivative notation as usual, $h^{(0)} = h$, $h' = h^{(1)}$, etc.*

(1) *we have*

$$\boxed{\mathcal{S}_n^{\vartheta}(x) = h_x^{(n)}(x)}; \tag{3.10}$$

(2) *putting*

$$T_x(t) = x + t \tag{3.11}$$

we have

$$\boxed{\mathcal{S}_n^{\vartheta,\, g_R}(x) = (h_x \circ T_x \circ g_R)^{(n)}(0)} . \tag{3.12}$$

Proof. Construct $\mathbf{F}$ as in (3.2) the fundamental pair associated to x with $f_{x,1}$ and $f_{x,2}$ to get $\mathbf{F}(x) = \mathbf{I}$, and thus

$$V(x, x+t) = h_x(t) . \tag{3.13}$$

Now, for (1), use Definition 3.1, and for Part (2), Definition 3.2. □

Remark 3.4. *From (3.10) and (3.12) note that since $h_x(x) = 0$, $h'_x(x) = 1$, $h''_x(x) = 0$, and, by (2.6), $2\,\vartheta(x) = h'''_x(x)$, we have $\mathcal{S}_0^{\vartheta} = 0$, $\mathcal{S}_1^{\vartheta} = 1$, $\mathcal{S}_2^{\vartheta} = 0$ and $\mathcal{S}_3^{\vartheta} = 2\,\vartheta$.*

Remark 3.5. *From the definition, each function $\mathcal{S}_n^{\vartheta,\, g_R}$ is a linear combination with constant coefficients of the Schwarzians; we have*

$$\boxed{\mathcal{S}_n^{\vartheta,\, g_R}(x) = \sum_{k=1}^{n} g_{R,\, n,\, k}\, \mathcal{S}_k^{\vartheta}(x)} \tag{3.14}$$

where, by Faà di Bruno formula,

$$\mathrm{g}_{R,\, n,k} = \overline{\sum}_{n,k} \prod_{i=1}^{n} \frac{1}{k_i!} \left(\frac{(\mathrm{g}_R)^{(i)}(0)}{i!} \right)^{k_i} \tag{3.15}$$

with $\overline{\Sigma}_{n,\, k}$ indicating the sum according to $\sum_{i=1}^{n} k_i = k$ and $\sum_{i=1}^{n} i\, k_i = n$.

An explicit computation of these coefficients for the choice $g_R(z) = -\dfrac{R}{\pi} \ln\left(1 - \dfrac{\pi\, z}{R}\right)$ is provided in Proposition 9.1 .

Remark 3.6. *From (3.14) it follows that the real parameter $R > 0$ enters in any g_R-Schwarzian polynomially in R^{-1}, and thus R can be extended to mean a non-zero complex number. For example, for $0 \neq \lambda \in \mathbb{R}$ put*

$$\boxed{\mathcal{S}_n^{\sigma,\, g_{\sqrt{-1}\lambda}} := \mathcal{S}_n^{\sigma,\, g_R}\Big|_{R=\sqrt{-1}\lambda}} . \tag{3.16}$$

We will use these in connection to closed geodesic in Section **??**

Definition 3.4. *Given $R \in (0, \infty]$, g_R as above, and integer $n \geq 1$ put*

$$\boxed{\mathcal{D}_n^{\vartheta,\, g_R}(x) := \det \left[\frac{\mathcal{S}_{i+j-1}^{\vartheta,\, g_R}(x)}{(i+j-1)!} \right]_{i,\, j=1}^{n}} . \tag{3.17}$$

Remark 3.7. *For each $x \in \mathbb{R}$ is defined and each integer $n \geq 1$,*

$$\lim_{R \to \infty} \mathcal{S}_n^{\vartheta,\, g_R}(x) = \mathcal{S}_n^{\vartheta}(x) , \tag{3.18}$$

$$\lim_{R \to \infty} \mathcal{D}_n^{\vartheta,\, g_R}(x) = \mathcal{D}_n^{\vartheta}(x) := \det \left[\frac{\mathcal{S}_{i+j-1}^{\vartheta}(x)}{(i+j-1)!} \right]_{i,\, j=1}^{n} . \tag{3.19}$$

4. Theorem 1. Continuation to a strip, finite or infinite

In this section we obtain a set of necessary and sufficient conditions on $\vartheta\colon \mathbb{R}\to\mathbb{R}$ so that it belongs to $\mathfrak{L}_R$.

The underlying result here is *Loewner's Theorem. A C^∞ function F on $(a,b)\subset\mathbb{R}\subset\mathbb{C}$ has the property that for all $n=1,2,\ldots$ and all $x\in(a,b)$*

$$\det\left[\frac{F^{(i+j-1)}(x)}{(i+j-1)!}\right]_{i,j=1}^{n} \geq 0 \tag{4.1}$$

if and only if it extends to a function on $(\mathbb{C}\setminus\mathbb{R})\cup(a,b)$ with positive imaginary part when $\Im z>0$, with equality true for a value of n if and only if F is rational.

In the proof of the next result, we will show the necessary condition using Fatou representation for Pick functions. As for the sufficient condition stated above, with the hypothesis at just one point and real analyticity to it, we adopt an argument from Bendat and Sherman [6].

Theorem 1. *Let $\vartheta\colon\mathbb{R}\to\mathbb{R}$ be smooth. Let $f_1=f_1(t)$ and $f_2=f_2(t)$ be any pair of independent solutions of $f''(t)+\vartheta(t)\,f(t)\;=\;0$. Put $W(f_1,\,f_2)\;:=\;f_1\,f_2'-f_1'\,f_2$ (a constant $\neq 0$) and set $h\;=\;f_2/f_1$. Let $R>0$ finite or infinite be given and take some map g_R as in (3.6).*

(1) *If h extends meromorphically on $\{\,z\in\mathbb{C}\,,|\Im(z)|\,<\,R\}$, analytically on* [2]

$$H_+^R\;:=\;\{\,z\in\mathbb{C}\,,0\,<\,\Im(z)\,<\,R\,\}\,, \tag{4.2}$$

with $W(f_1,\,f_2)\,\Im h(z)\,\Im(z)>0$, then, ϑ is real analytic on $\mathbb{R}$, and for all integer $n\geq 1$

$$\mathcal{D}_n^{\vartheta,\,g_R}\;\geq\;0\,.$$

(2) *If ϑ is real analytic on $\mathbb{R}$ and there is $x_0\,\in\,\mathbb{R}$ such that for all integer $n\,\geq\,1$*

$$\mathcal{D}_n^{\vartheta,\,g_R}(x_0)\;\geq\;0$$

then h extends as in (1).

Proof. For Part (1). On account that

(i) quotients of independent solutions are related by real Moëbius transformations,

(ii) a Moëbius transformation $M(z)\;=\;\dfrac{a\,z+b}{c\,z+d}$ with $a,b,c,d\in\mathbb{R}$ sends H_+^R to itself if and only if $\det M\;=\;a\,d-c\,d\;>\;0$, and

(iii) $W(af_1+bf_2,cf_1+df_2)\;=\;W(f_1,f_2)\,\det M$,

it follows that a map h satisfies the hypothesis in 1) if and only any real Moëbius transformation of h does.

Consequently, if x is any point of $\mathbb{R}$, the map $h_x\;=\;\dfrac{f_{x,2}}{f_{x,1}}$, which is the quotient of the pair of fundamental solutions associated to x (Definition 3.3), satisfies hypothesis 1). Since $h_x'(x)\;=\;1\neq 0$, the classical Schwarzian of h_x computes $2\,\vartheta$ and shows its real analyticity, in some neighborhood of x. It follows the analyticity in all $\mathbb{R}$.

Now, let $R\in(0,\infty]$ be given and again let x be any point of $\mathbb{R}$. Consider the composition

$$h_x\circ T_x\circ g_R \tag{4.3}$$

with T_x as in (3.11) and g_R as in (3.6). Since g_R is a Pick function defined in $(\mathbb{C}\,\setminus\,\mathbb{R})\cup(a,b)$, with $a\,<\,0\,<\,b$, and maps into $|\Im(z)|\,<\,R$, there is an $\epsilon_x\;=\;\epsilon_x^{\vartheta,\,g_R}\;>\;0$ such that (4.3) is defined and holomorphic on

$$H_+\,\cup\,H_-\,\cup\,(-\epsilon_x,\,\epsilon_x)\;. \tag{4.4}$$

Moreover, the map (4.3) has positive imaginary part on H_+ since g_R and h_x both do (g_R by definition, and h_x since it satisfies (1) and $W(f_{x,1},f_{x,2})\;=\;1\;>\;0$). So, the map (4.3) is a Pick function with domain (4.4).

According to the Fatou representation for a Pick function, for $z\in\mathbb{C}$,

$$(h_x\circ T_x\circ g_R)\,(z)\;=\;\alpha_x\,z\;+\;\beta_x\;+\int_{-\infty}^{\infty}\left(\frac{1}{t-z}-\frac{t}{1+t^2}\right)\,d\nu_x(t), \tag{4.5}$$

[2] For $R\;=\;\infty$ (4.2), the upper-half plane, is denoted by H_+.

with constants

$$\alpha_x = \alpha_x^{\vartheta, g_R} \geq 0, \quad \beta_x = \beta_x^{\vartheta, g_R} \in \mathbb{R},$$

and with $d\nu_x = d\nu_x^{\vartheta, g_R}$ a non-negative Borel measure with $\int_{-\infty}^{\infty} \frac{d\nu_x(t)}{1+t^2} < \infty$.

With $z = x + \sqrt{-1}y$, $x, y \in \mathbb{R}$, the measure $d\mu_x$ is given by the weak limit of

$$\Im\left(h_x \circ T_x \circ g_R\right)(x + \sqrt{-1}y) \quad \text{as} \quad y \mapsto 0^+ .$$

Since the map $h_x \circ T_x \circ g_R$ is real valued in $(-\epsilon_x, \epsilon_x) \subset \mathbb{R}$ we have

$$\int_{-\epsilon_x}^{\epsilon_x} d\mu_x(t) = \lim_{y \mapsto 0^+} \int_{-\epsilon_x}^{\epsilon_x} \Im\left(h_x \circ T_x \circ g_R\right)(t + \sqrt{-1}y)\, dt = 0 .$$

Thus

$$\text{Support}\,(d\nu_x) \cap (-\epsilon_x, \epsilon_x) = \emptyset\,, \tag{4.6}$$

and we can take derivatives at the origin under the integral sign to get

$$1 \equiv \mathcal{S}_1^{\vartheta, g_R}(x) = \left(h_x \circ T_x \circ g_R\right)'(0) = \alpha_x + \int_{-\infty}^{\infty} \frac{d\nu_x(t)}{t^2}\,, \tag{4.7}$$

and for all $n \geq 2$

$$\mathcal{S}_n^{\vartheta, g_R}(x) = \left(h_x \circ T_x \circ g_R\right)^{(n)}(0) = n! \int_{-\infty}^{\infty} \frac{d\nu_x(t)}{t^{n+1}}\,. \tag{4.8}$$

It follows that

$$\det \left[\frac{\mathcal{S}_{i+j-1}^{\vartheta, g_R}(x)}{(i+j-1)!} \right]_{i,j=1}^{n} \tag{4.9}$$

is represented by

$$\det \begin{bmatrix} \alpha_x + \int_{-\infty}^{\infty} t_1^{-2} d\nu_x(t_1) & \int_{-\infty}^{\infty} t_2^{-3} d\nu_x(t_2) & \cdots & \int_{-\infty}^{\infty} t_n^{-n-1} d\nu_x(t_n) \\ \int_{-\infty}^{\infty} t_1^{-3} d\nu_x(t_1) & \int_{-\infty}^{\infty} t_2^{-4} d\nu_x(t_2) & \cdots & \int_{-\infty}^{\infty} t_n^{-n-2} d\nu_x(t_n) \\ \cdots & \cdots & \cdots & \cdots \\ \int_{-\infty}^{\infty} t_1^{-n-1} d\nu_x(t_1) & \int_{-\infty}^{\infty} t_2^{-n-2} d\nu_x(t_2) & \cdots & \int_{-\infty}^{\infty} t_n^{-2n} d\nu_x(t_n) \end{bmatrix}.$$

Due to the symmetry of the Hankel matrix, the multi-linearity of the determinant, and the linear nature of integration, the expression above is written as

$$\begin{aligned} &\alpha_x \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \det \left[t_j^{-i-j} \right]_{i,j=2}^{n} d\nu_x(t_2) \cdots d\nu_x(t_n) + \\ &+ \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \det \left[t_j^{-i-j} \right]_{i,j=1}^{n} d\nu_x(t_1) \cdots d\nu_x(t_n)\,. \end{aligned} \tag{4.10}$$

Consider the integrand corresponding to the second term in (4.10); it can be written as

$$\det \left[t_j^{-i-j} \right]_{i,j=1}^{n} = \det \left[t_j^{-i} \right]_{i,j=1}^{n} \prod_{j=2}^{n} t_j^{-j}. \tag{4.11}$$

Let Σ be the permutation group of the set $\{1, \cdots, n\}$. For any $\delta \in \Sigma$, from (4.11),

$$\begin{aligned} \det \left[t_{\delta(j)}^{-i-j} \right]_{i,j=1}^{n} &= \det \left[t_{\delta(j)}^{-i} \right]_{i,j=1}^{n} \prod_{j=1}^{n} t_{\delta(j)}^{-j} \\ &= \text{sign}(\delta) \det \left[t_j^{-i} \right]_{i,j=1}^{n} \prod_{j=1}^{n} t_{\delta(j)}^{-j}\,, \end{aligned}$$

from which it follows that

$$\sum_{\delta \in \Sigma} \det \left[t_{\delta(j)}^{-i-j} \right]_{i,j=1}^{n} = \left(\det \left[t_j^{-i} \right]_{i,j=1}^{n} \right)^2 . \tag{4.12}$$

Now, since no $\delta \in \Sigma$ influences the value of

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \det\left[t_{\delta(j)}^{-i-j}\right]_{i,j=1}^{n} d\nu_x(t_1)\cdots d\nu_x(t_n)\,,$$

by integrating (4.12), the second integral in (4.10) equals

$$\frac{1}{n!}\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left(\det\left[t_j^{-i}\right]_{i,j=1}^{n}\right)^2 d\nu_x(t_1)\cdots d\nu_x(t_n)\,.$$

The argument for the integral in (4.10) corresponding to α_x is similar.

In conclusion,

$$\begin{aligned} \text{(4.13)}\quad \mathcal{D}_n^{\vartheta,g_R}(x) &= \det\left[\frac{\mathcal{S}_{i+j-1}^{\vartheta,g_R}(x)}{(i+j-1)!}\right]_{i,j=1}^{n} \\ &= \frac{\alpha_x}{(n-1)!}\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left(\det\left[t_j^{-i}\right]_{i,j=2}^{n}\right)^2 d\nu_x(t_2)\cdots d\nu_x(t_n) + \\ &\quad + \frac{1}{n!}\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left(\det\left[t_j^{-i}\right]_{i,j=1}^{n}\right)^2 d\nu_x(t_1)\cdots d\nu_x(t_n)\,, \end{aligned}$$

which is non-negative, due to the non-negativity of α_x and of the measure.

Since $x \in \mathbb{R}$ was arbitrary, Part (1) is proved.

For Part (2).

By the opening paragraph of the proof of Part (1) it suffices to consider the case $h_{x_0} = f_{x_0,2}/f_{x_0,1}$, the quotient of the fundamental solutions associated to x_0 as in (3.9).

From the hypothesis on ϑ, h_{x_0} is real analytic on some neighborhood of x_0 and thus so is $h_{x_0}\circ T_{x_0}\circ g_R$ in some neighborhood of the origin. Thus for all integer $n \geq 1$,

$$\text{(4.14)}\qquad \det\left[\frac{(h_{x_0}\circ T_{x_0}\circ g_R)^{(i+j-1)}}{(i+j-1)!}(0)\right]_{i,j=1}^{n} = \mathcal{D}_n^{\vartheta,R}(x_0) \geq 0\,.$$

Here the equality is valid by Proposition 3.1 together with $(h_{x_0}\circ T_{x_0}\circ g_R)^{(n)}(0) = \mathcal{S}_n^{\vartheta,g_R}(x_0)$, while the inequality holds by hypothesis.

Claim: *From (4.14), it follows that $h_{x_0}\circ T_{x_0}\circ g_R$ extends as a Pick function.*

Proof of Claim. The claim is a result by Bendat and Sherman [6] plus compositions of functions that we want to keep track of. We rely on the solution of the classical Hamburguer moment problem [27] which given the non-negativity of the Hankel determinants guarantees the existence of a non-decreasing function

$$\text{(4.15)}\qquad \mu_{x_0} = \mu_{x_0}^{\vartheta,g_R} : \mathbb{R} \to \mathbb{R}$$

such that in terms of Stieltjes integrals,

$$\text{(4.16)}\qquad (h_{x_0}\circ T_{x_0}\circ g_R)^{(n)}(0) = n!\int_{-\infty}^{\infty} t^{n-1} d\mu_{x_0}(t)\,.$$

Then, by the real analyticity $h_{x_0}\circ T_{x_0}\circ g_R$ in a neighborhood of $x = 0$,

$$\begin{aligned} \text{(4.17)}\qquad F_{x_0}^{\vartheta,g_R}(z) &:= \sum_{n=1}^{\infty} (h_{x_0}\circ T_{x_0}\circ g_R)^{(n)}(0)\,\frac{z^n}{n!} \\ &= \sum_{n=1}^{\infty}\left(\int_{-\infty}^{\infty} t^{n-1} d\mu_{x_0}(t)\right) z^n\,, \end{aligned}$$

is absolutely convergent for $z \in \mathcal{O}_{x_0} = \mathcal{O}_{x_0}^{\vartheta,g_R} = \{z \in \mathbb{C},\ |z| < \rho_{x_0}\}$, where

$$\text{(4.18)}\qquad \rho_{x_0} = \rho_{x_0}^{\vartheta,g_R} > 0$$

is the radius of convergence of $h_{x_0}\circ T_{x_0}\circ g_R$.

The measure $d\mu_{x_0}$ is supported in the interval

$$[-1/\rho_{x_0},\, 1/\rho_{x_0}] \subset \mathbb{R}\,,$$

since otherwise, there is a constant $C > 1/\rho_{x_0}$ such that for all integer $m \geq 1$

$$\int_{-\infty}^{\infty} t^{2m}\, d\mu_{x_0}(t) \;\geq\; C^{2m} \left(\int_{-\infty}^{-C} d\mu_{x_0}(t) + \int_{C}^{\infty} d\mu_{x_0}(t) \right) \;>\; 0\,,$$

and thus $\lim_{m\mapsto\infty} \sqrt[2k+1]{\int_{-\infty}^{\infty} t^{2m}\, d\mu_{g_R,x_0}} \;\geq\; C \;>\; 1/\rho_{x_0}$, which would imply a radius of convergence of $h_{x_0} \circ T_{x_0} \circ g_R$ strictly smaller that ρ_{x_0}, a contradiction.

So we now interchange summations in

$$F_{x_0}^{\vartheta,\,g_R}(z) \;=\; \sum_{n=1}^{\infty} \left(\int_{-1/\rho_{x_0}}^{1/\rho_{x_0}} t^{n-1} d\mu_{x_0}(t) \right) z^n\,, \tag{4.19}$$

to obtain for $z \in \mathcal{O}_{x_0}$

$$F_{x_0}^{\vartheta,\,g_R}(z) \;=\; \int_{-1/\rho_{x_0}}^{1/\rho_{x_0}} \frac{z}{1 - t\,z}\, d\mu_{x_0}(t)\,. \tag{4.20}$$

Since, by construction, on the interval $(-\rho_{x_0},\, \rho_{x_0}) \;=\; \mathcal{O}_0 \cap \mathbb{R}$ we have the agreement

$$F_{x_0}^{\vartheta,\,g_R} \;\equiv\; h_{x_0} \circ T_{x_0} \circ g_R\,,$$

the function (4.20) represents a holomorphic extension of $h_{x_0} \circ T_{x_0} \circ g_R$ which is now defined on

$$(\mathbb{C} \setminus \mathbb{R}) \cup \mathcal{O}_0\,,$$

and that maps H_+ to itself. This is the extension as a Pick function derived from the conditions (4.14) whose existence was claimed. □

Now, from this, the function

$$\widehat{h}_{x_0} \;:=\; F_{x_0}^{\vartheta,\,g_R} \circ g_R^{-1} \circ T_{-x_0} \tag{4.21}$$

is holomorphic on

$$H_+^R \,\cup\, H_-^R \,\cup\, \mathcal{U}_{x_0}, \tag{4.22}$$

where $\mathcal{U}_{x_0} = T_{x_0}\,(g_R\,(\mathcal{O}_0))$ is an open neighborhood of x_0 in $\mathbb{C}$, and maps H_+^R to H_+.

This means that $\widehat{h}_{x_0}$ provides an extension of h_{x_0} as a Pick function with domain (4.22). We now check that it extends across $\mathbb{R} \setminus f_{x_0,\,1}^{-1}(0)$ as well, as a Pick function, which will complete the proof.

To see this last point, note that, since the original h_{x_0} is real analytic on $\mathbb{R} \setminus f_{x_0,\,1}^{-1}(0)$, there is an open set $\mathcal{U}$ of $\mathbb{C}$, with

$$x_0 \,\in\, \mathbb{R} \setminus f_{x_0,\,1}^{-1}(0) \,\subset\, \mathcal{U}\,, \tag{4.23}$$

and a holomorphic function

$$\widetilde{h}_{x_0} \colon \mathcal{U} \to \mathbb{C}$$

so that on $\mathbb{R} \setminus f_{x_0,\,1}^{-1}(0)$,

$$\widetilde{h}_{x_0} \;\equiv\; h_{x_0}\,.$$

In particular, $\widetilde{h}_{x_0}$ is determined near x_0 by the Taylor series of h_{x_0} centered at x_0, which has a non-zero radius of convergence, say r_0. So, on the neighborhood of x_0 in $\mathbb{C}$ given by

$$\{z \,\in\, \mathbb{C},\, |z - x_0| < r_0\} \,\cap\, \mathcal{O}_0\,,$$

we have the coincidence

$$\widetilde{h}_{x_0} \;\equiv\; \widehat{h}_{x_0}\,. \tag{4.24}$$

Thus, by uniqueness of analytic continuation, the identity (4.24) also holds in

$$\mathcal{U} \cap H_+^R\,,$$

and, similarly, in

$$\mathcal{U} \cap H_-^R\,.$$

This means, in light of (4.23), that $\widetilde{h}_{x_0}$ provides an analytic extension of $\widehat{h}_{x_0}$ across the set $\mathbb{R}\setminus f_{x_0,1}^{-1}(0)$, where the function $\widetilde{h}_{x_0}$ assumes only real values. It follows that $\widehat{h}_{x_0}$ is now analytically extended as a Pick function with domain

$$H_+^R \cup H_-^R \cup \mathbb{R}\setminus f_{x_0,1}^{-1}(0) \;=\; \mathbb{C}\setminus f_{x_0,1}^{-1}(0)\,.$$

It extends meromorphically on $\mathbb{C}$ with (simple) poles at $f_{x_0,1}^{-1}(0)\subset\mathbb{R}$.

This proves part (2). □

Remark 4.1. *Theorem 1 shows that, for ϑ real analytic in $\mathbb{R}$, if there is an $x_0\in\mathbb{R}$ so that for all $n\geq 1$ $\mathcal{D}_n^{\vartheta,g_R}(x_0)\geq 0$ then $\mathcal{D}_n^{\vartheta,g_R}(x)\geq 0$ for all $n\geq 1$ and for all $x\in\mathbb{R}$. This will be used in Corollary 6.0.1.*

4.1. **A simple curvature estimate for the measure.**

We make a few comments on the relation exploited in the proof of Theorem 1, of the Schwarzians on $\mathbb{R}$ and moments.

Recall from the proof of part (2) of Theorem 1 its equivalence to the possibility of the Schwarzians being point-wise expressible as moments

$$\boxed{\mathcal{S}_n^{\vartheta,g_R}(x) \;=\; n!\int_{-1/\rho_x}^{1/\rho_x} t^{n-1}\, d\mu_x(t)}\,, \tag{4.25}$$

where, for each x, $d\mu_x(t)=d\mu_x^{\vartheta,g_R}(t)$ is a non-negative Borel measure in R with

$$\text{Support}\, d\mu_x(t) \;\subset\; [-1/\rho_x,\; 1/\rho_x] \tag{4.26}$$

for some

$$\rho_x \;=\; \rho_x^{\vartheta,g_R} \;\in\; (0,\,\infty]\,.$$

There is a very simple estimate for the support of that measure in terms of curvature as follows.

Proposition 4.1. *Assume ϑ is as in part (1) of Theorem 1. Take $g_R(z)\;=\;-\dfrac{R}{\pi}\ln\left(1-\dfrac{\pi z}{R}\right)$. Set $\sup\vartheta\;=\;\sup_{x\in\mathbb{R}}\vartheta$. Then for any $x\in\mathbb{R}$,*

1. *if* $\sup\vartheta\leq 0$, Support $\left(d\mu_x^{\vartheta,g_R}\right)\;\subset\;\left[-\dfrac{\pi}{R}\,,\;\dfrac{\pi}{R}\right]$, *interpreted as the set* $\{0\}$ *if* $R\;=\;\infty$;
2. *if* $\sup\vartheta\;>\;0$, Support $\left(d\mu_x^{\vartheta,g_R}\right)\;\subset\;\left[-\dfrac{\pi}{R-\,R\,e^{-\frac{\pi^2}{2R\sqrt{\sup\vartheta}}}}\,,\;\dfrac{\pi}{R-\,R\,e^{-\frac{\pi^2}{2R\sqrt{\sup\vartheta}}}}\right]$,
   interpreted as the interval $\left[-\dfrac{2\sqrt{\sup\vartheta}}{\pi}\,,\;\dfrac{2\sqrt{\sup\vartheta}}{\pi}\right]$ *if* $R\;=\;\infty$.

Proof. Let $x\in\mathbb{R}$ be arbitrary. With notation as in the proof of Theorem 1, for some $\rho_x>0$ the series in of z of $h_x\circ T_x\circ g_R$ is absolutely convergent for $|z|\;<\;\rho_x$, and necessarily, given our choice of g^R,

$$(-\rho_x,\,\rho_x)\;\subset\;\left(\frac{R}{\pi}\left(1-e^{\frac{\pi}{R}\epsilon_1(x)}\right),\,\frac{R}{\pi}\left(1-e^{-\frac{\pi}{R}\,\epsilon_2(x)}\right)\right)\;\subset\;\left(-\infty,\,\frac{R}{\pi}\right)$$

with $\epsilon_1(x)\;:=\;\sup_{t>0}\{\,t\,|\,(x-t,\,x)\subset\mathbb{R}\setminus f_1^{-1}(0)\}$, $\epsilon_2(x)\;:=\;\sup_{t>0}\{\,t\,|\,(x,\,x+t)\subset\mathbb{R}\setminus f_1^{-1}(0)\}$. Thus, we may take

$$\begin{aligned}
\rho_x \;&=\; \min\left\{\frac{R}{\pi}\left(e^{\frac{\pi}{R}\epsilon_1(x)}-1\right),\,\frac{R}{\pi}\left(1-e^{-\frac{\pi}{R}\,\epsilon_2(x)}\right)\right\}\\
&\overset{(i)}{\geq}\; \min\left\{\frac{R}{\pi}\left(1-e^{-\frac{\pi}{R}\epsilon_1(x)}\right),\,\frac{R}{\pi}\left(1-e^{-\frac{\pi}{R}\,\epsilon_2(x)}\right)\right\}\\
&\overset{(ii)}{=}\; \frac{R}{\pi}\left(1-e^{-\frac{\pi}{R}\min\{\epsilon_1(x),\,\epsilon_2(x)\}}\right)\\
&\overset{(iii)}{\geq}\; \begin{cases}\frac{R}{\pi}\;\text{ if }\vartheta\;\leq\;0\,,\\[2ex] \dfrac{R}{\pi}\left(1-e^{-\frac{\pi^2}{2R\sqrt{\sup\vartheta}}}\right)\;\text{ if }\sup\vartheta\;>\;0\,,\end{cases}
\end{aligned}$$

where (i) is valid since $e^t - 1 > 1 - e^{-t}$ for $t > 0$; (ii) holds because $1 - e^{-t}$ is an increasing function; inequality (iii) follows from a standard Sturm comparison of the location of zeros of solutions of $f''(t) + \vartheta(t)\, f(t) = 0$, with those of $\cos(\sqrt{K}(t-x))$, with $K = \sup \vartheta$.

Now, use 4.26 and vary x_0. □

5. Higher order Schwarzians and related functions in the tangent bundle

The higher order Schwarzians $\mathrm{S}_n^{\vartheta, R}$ defined on $\mathbb{R}$ have canonical expressions in ϑ and its derivatives which Tamanoi proved [25] (See Remark 5.1). This will be the basis for our definition of the functions $\mathrm{S}_n^{\sigma, R}$ in the tangent bundle .

We will re-derive Tamanoi's expressions in the spirit of the last section in reference [17], using Proposition 3.1 and the equation (3.1) directly, in order to keep track of the weight w introduced below used to describe homogeneity properties of the functions $\mathrm{S}_n^{\sigma, R}$.

5.1. Schwarzians in terms of curvature via Tamanoi polynomials.

Definition 5.1. *A monomial in* $\{\vartheta, \vartheta', \ldots, \vartheta^{(r)}, \ldots\}$ *is said to have weight* w *according to*

$$\mathrm{w}\left\{\prod_{i=1}^{n}\left(\vartheta^{(r_i)}(t)\right)^{l_i}\right\} = \sum_{i=1}^{n}(2+r_i)\, l_i.$$

A polynomial all of whose terms have the same w *will be called* isobaric.

Proposition 5.1. *Let* $m = \prod_{i=1}^{n}\left(\vartheta^{(r_i)}\right)^{l_i}(t)$ *be a monomial in* ϑ *and its derivatives. Then*
(i) $\mathrm{w}\{\vartheta(t)\, m(t)\} = \mathrm{td}\{m\} + 2$ *;*
(ii) $\mathrm{w}\{m'(t)\} = \mathrm{w}\{m\} + 1$ *.*

Proof. It is enough to show this for $m = \left(\vartheta^{(r)}\right)^l$. Now, (i) is clear since $\mathrm{w}\{\vartheta\} = 2$. For (ii), note $m' = l\left(\vartheta^{(r)}\right)^{l-1}\vartheta^{(r+1)}$ and hence

$$\mathrm{w}\{m'\} = (2+r)(l-1) + (2+r+1) = (2+r)\,l + 1 = \mathrm{w}\{m\} + 1\,.$$

□

We note, $\mathrm{w}\{\vartheta\} = 2$, $\mathrm{w}\{\vartheta'\} = 3$, $\mathrm{w}\{\vartheta''\} = 4$, $\mathrm{w}\{(\vartheta')^2\} = \mathrm{w}\{\vartheta^{(4)}\} = 6$, hence the Schwarzian $\mathcal{S}_7^{\vartheta} = \vartheta^{(4)} + 76\,\vartheta''\vartheta + 52\,(\vartheta')^2 + 272\,\vartheta^3$ is isobaric with weight w equal to 6.

In general, we have the following

Proposition 5.2. *The Schwarzian* $\mathcal{S}_n^{\vartheta}$ *can be expressed as a polynomial in* ϑ *and its derivatives up to order* $n-3$*,*

$$\pi_n(\vartheta,\, \vartheta',\, \ldots,\, \vartheta^{(n-3)}), \tag{5.1}$$

which is isobaric of weight w *equal to* $n-1$*.*

Proof. For convenience of reference let us recall here that by Proposition 3.1, at a point x

$$\mathcal{S}_n^{\vartheta}(x) = h_x^{(n)}(x) \tag{5.2}$$

where $h_x(t) = \frac{f_{x,2}(t)}{f_{x,1}(t)}$, with $f_{x,1}$ and $f_{x,2}$ as in Definition 3.3, that is, they are the solutions of

$$f''(t) + \vartheta(t) f(t) = 0 \tag{5.3}$$

with the initial conditions so that

$$f_{x,1}(x) = f'_{x,2}(x) = h'_x(x) = 1, \quad f_{x,2}(x) = f'_{x,1}(x) = h_x(x) = h''_x(x) = 0. \tag{5.4}$$

Now, (5.3) and two derivatives with respect to t of $h_x(t)\, f_{x,1}(t) = f_{x,2}(t)$ gives

$$2\, h'_x(t)\, f'_{x,1}(t) + h''_x(t)\, f_{x,1}(t) = 0. \tag{5.5}$$

By use of (5.3) the n-th derivative with respect to t of the left-hand side of (5.5) is written as

$$\beta_{x,n}(t)\, f'_{x,1}(t) + \alpha_{x,n}(t)\, f_{x,1}(t) = 0\,, \tag{5.6}$$

where for $n = 1, 2, \cdots$ the coefficients of $f_{x,1}(t)$ and $f'_{x,1}(t)$ are computed by

$$\begin{pmatrix} \alpha_{x,n}(t) \\ \beta_{x,n}(t) \end{pmatrix} = \begin{pmatrix} \partial_t & -\vartheta(t) \\ 1 & \partial_t \end{pmatrix}^n \begin{pmatrix} h''_x(t) \\ 2\, h'_x(t) \end{pmatrix}$$

with $(\partial_t)^k y$ meant to indicate $y^{(k)}$.

Consider as inductive hypothesis that for $n \in \mathbb{N}$ it is true that

$$\alpha_{x,\,n}(t) \;=\; h_x^{(n+2)}(t) + p_{x,\,n}(t), \tag{5.7}$$

$$\beta_{x,\,n}(t) \;=\; h_x^{(n+1)}(t) + q_{x,n}(t), \tag{5.8}$$

where both $p_{x,\,n}(t)$ and $q_{x,\,n}(t)$ are integer linear combinations of terms of the form

$$h_x^{(k)}(t)\prod_{i=1}\left(\vartheta^{(r_i)}(t)\right)^{l_i} \tag{5.9}$$

with their indices k, r_i and l_i restricted for the terms in $p_{x,\,n}(t)$ by

$$k\,+\,\sum_i(2+r_i)\,l_i \;=\; n+2\,, \qquad k\le n+1\,, \tag{5.10}$$

while for the terms in $q_{x,\,n}(t)$ those indices are restricted by

$$k\,+\,\sum_i(2+r_i)\,l_i \;=\; n+1\,, \qquad k\le n\,. \tag{5.11}$$

Since $\alpha_{x,\,1}(t) \;=\; h_x'''(t) - 2\vartheta(t)\,h_x'(t)$ and $\beta_{x,\,1}(t) \;=\; -h_x''(t)$ then (5.7) through (5.11) hold for $n=1$. Moreover, by Proposition 5.1 applied to

$$\alpha_{x,\,n+1}(t) = \alpha'_{x,\,n}(t) - \vartheta(t)\;\beta_{x,\,n}(t)\;, \quad \beta_{x,n+1}(t) = \beta'_{x,\,n}(t) + \alpha_{x,\,n}(t)$$

it follows that (5.7) through (5.11) are valid for $n+1$ and hence for all $n\in\mathbb{N}$.

Now make the inductive hypothesis "$\mathcal{S}^{\vartheta}_{n+1}$ is isobaric of weight w equal to n", which certainly holds for $n=0$, since $\mathcal{S}^{\vartheta}_1 = 1$. We have shown that for all n

$$\alpha_{x,n}(t) \;=\; h_x^{(n+2)}(t) + \sum a_{k,r_i,l_i}\,h_x^{(k)}(t)\prod_{i=1}\left(\vartheta^{(r_i)}(t)\right)^{l_i} \tag{5.12}$$

with a_{k,r_i,l_i} integers and with the indices k, r_i and l_i that satisfy (5.10). Thus since by (5.6) we have $\alpha_{x,n}(x)=0$ then putting $t=x$ in (5.12) gives

$$\mathcal{S}^{\vartheta}_{n+2}(x) \;=\; -\sum a_{k,r_i,l_i}\,\mathcal{S}^{\vartheta}_k(x)\prod_{i=1}\left(\vartheta^{(r_i)}(x)\right)^{l_i}\;. \tag{5.13}$$

But since according to (5.10) $k\le n+1$ using the inductive hypothesis for the $\mathcal{S}^{\vartheta}_k(x)$ we get that

$$\mathrm{w}\left\{\mathcal{S}^{\vartheta}_k(x)\prod_{i=1}\left(\vartheta^{(r_i)}(x)\right)^{l_i}\right\} = \mathrm{w}\left\{\mathcal{S}^{\vartheta}_k(x)\right\} + \mathrm{w}\left\{\prod_{i=1}\left(\vartheta^{(r_i)}(x)\right)^{l_i}\right\} = k-1+\sum_i(2+r_i)\,l_i \;=\; n+1,$$

the last equality of course in light of (5.10). Thus $\mathcal{S}^{\vartheta}_{n+2}$ is isobaric with weight w equal to $n+1$. □

Remark 5.1. *Let T_n be the Tamanoi polynomials* [25] *. Then*

$$\pi_n(x_1,\,\cdots\,,\,x_{n-3}) \;=\; T_{n-1}(2\,x_1,\,\cdots\,,\,2\,x_{n-3})$$

Accordingly our weight equals Tamanoi's "virtual order" plus one.

5.2. **Schwarzians in the tangent bundle.**

We transplant the previous construction to the context of Riemannian manifolds. Consider M to be a smooth manifold and $\pi\colon TM\to M$ its tangent bundle. Let g be a smooth Riemannian metric on M with ∇ the corresponding Levi-Civita connection and $\sigma\colon M\to\mathbb{R}$ the Gauss curvature; the pull-back

$$\pi^*\sigma \;=\; \sigma\circ\pi\colon TM\to\mathbb{R}$$

is also denoted by σ, to simplify notation. For any $\mathrm{v}\in TM$ we put $\|\mathrm{v}\|^2 \;=\; g(\mathrm{v},\mathrm{v})$.

Definition 5.2. *The higher order Schwarzians in TM,*

$$\mathrm{S}_n^{\sigma} \colon TM \to \mathbb{R}$$

for $n \geq 1$, are defined by performing in the Schwarzians on $\mathbb{R}$ from Proposition 5.2,

$$\mathcal{S}_n^{\sigma} \;=\; \pi_n\,(\sigma,\, \sigma',\, \ldots,\, \sigma^{(n-3)}\,)$$

the substitutions

$$\sigma^{(k)} \mapsto \|\mathrm{v}\|^2\, \nabla_{\mathrm{v}}^{k}\sigma\,. \tag{5.14}$$

That is, at $\mathrm{v} \in TM$,

$$\mathrm{S}_n^{\sigma}(\mathrm{v}) \;:=\; \pi_n\,(\,\|\mathrm{v}\|^2\;\sigma,\; \|\mathrm{v}\|^2\;\nabla_{\mathrm{v}}\sigma,\; \ldots,\; \|\mathrm{v}\|^2\;\nabla_{\mathrm{v}}^{n-3}\sigma\,)\,. \tag{5.15}$$

Remark 5.2. *The justification of the factor $\|\mathrm{V}\|^2$ is given in Remark 6.2.*

Definition 5.3. *Given $R \in (0,\,\infty\,]$ and g_R as in (3.6) for $n \geq 1$ the functions*

$$\mathrm{S}_n^{\sigma,\,g_R} \colon TM \to \mathbb{R}$$

are defined by

$$\boxed{\mathrm{S}_n^{\sigma,\,g_R}(\mathrm{v}) \;=\; \sum_{k=1}^{n} g_{R,\,n,\,k}\; \|\mathrm{v}\|^{n-k}\; \mathrm{S}_k^{\sigma}(\mathrm{v})}\,, \tag{5.16}$$

with constants $g_{R,\,n,\,k}$ according to (5.16) and with S_n^{σ} given by (5.15).

Remark 5.3. *It follows by Remark 3.7 that,*

$$\boxed{\mathrm{S}_n^{\sigma,\,g_R} \text{ reduce in the limit } R \,=\, \infty \text{ to } \mathrm{S}_n^{\sigma}}\,.$$

Remark 5.4. *If M is the zero section of TM, then from (5.16),*

$$\boxed{\mathrm{S}_1^{\sigma,\,g_R}\,|_M \;\equiv 1,\;\;\; \mathrm{S}_n^{\sigma,\,g_R}\,|_M \;\equiv 0,\; n \neq 1}$$

.

For higher n and $g_R(z) \;=\; -\dfrac{R}{\pi}\,\ln\left(1 - \dfrac{\pi\, z}{R}\right)$ see Proposition 9.1, Corollary 9.1.1 .

Definition 5.4. *Given $R \in (0,\,\infty]$, for $\mathrm{v} \in TM$ and integer $n \geq 1$ put*

$$\mathrm{D}_n^{\sigma,\,g_R}(\mathrm{v}) \quad := \quad \det\left[\frac{\mathrm{S}_{i+j-1}^{\sigma,\,g_R}(\mathrm{v})}{(i+j-1)!}\right]_{i,\,j\,=\,1}^{n}\,. \tag{5.17}$$

The following is used in the first part of the proof of Theorem 2, in computations and in Section 7.

Proposition 5.3. *The functions $\mathrm{S}_n^{\sigma,\,g_R}$ and $\mathrm{D}_n^{\sigma,\,g_R}$ are homogeneous along the fibers of TM of degree $n-1$ and $n(n-1)$ respectively. They are all real analytic if g is.*

Proof. Given the integer $k \geq 0$, by the linearity of ∇ in its lower argument, the function $\|\mathrm{v}\|^2\;\nabla_{\mathrm{v}}^{k}\sigma$ is homogeneous of degree $2+k$ along the fibers of TM; thus

$$\text{degree of } \prod_{i=1}^{n}(\|\mathrm{v}\|^2\;\nabla_{\mathrm{v}}^{k_i}\sigma)^{l_i} \;=\; \sum_{i=1}^{n}(2+k_i)\,l_i \;=\; \mathrm{w}\left\{\prod_{i=1}^{n}\left(\sigma^{(k_i)}\right)^{l_i}\right\},$$

where for the second equality we have used $2+k \;=\; \mathrm{w}\{\sigma^{(k)}\}$ by Definition 5.1.

Pair this with (5.15) and the fact the polynomials π_n are isobaric with $\mathrm{w}\{\pi_n\} \;=\; n-1$ by Proposition 5.2, and we have that for $\lambda \neq 0$,

$$\mathrm{S}_n^{\sigma}(\lambda\,\mathrm{v}) \;=\; \lambda^{n-1}\,\mathrm{S}_n^{\sigma}(\mathrm{v})\,,$$

which used in (5.16) shows the homogeneity property claimed,

$$\mathrm{S}_n^{\sigma,\,g_R}(\lambda\,\mathrm{v}) \;=\; \lambda^{n-1}\,\mathrm{S}_n^{\sigma,\,g_R}(\mathrm{v})\,. \tag{5.18}$$

In light of this, it follows that for every non-negative integer $n \geq 2$ and any $\mathrm{v} \neq 0$

$$\det\left[\frac{\mathrm{S}^{\sigma,\, g_R}_{(i+j-1)}(\lambda\,\mathrm{v})}{(i+j-1)!}\right]^n_{i,\,j\,=\,1} \;=\; \lambda^N \det\left[\frac{\mathrm{S}^{\sigma,\, g_R}_{(i+j-1)}(\mathrm{v})}{(i+j-1)!}\right]^n_{i,\,j\,=\,1}\,, \tag{5.19}$$

where $N = n(n-1)$; indeed, for a given n, the determinant that gives $\mathrm{D}^{\sigma,\, g_R}_n$ consists of a sum of terms of the form $\prod_{i=1}^{n} \mathrm{S}^{\sigma,\, g_R}_{(i+\pi(\,i\,)-1)}(\mathrm{v})$, for some permutation p of $\{\,1,\,\ldots,\,n\,\}$, and hence, by (5.18), of common degree

$$N \,=\, \sum_{i\,=\,1}^{n} i + \mathrm{p}(\,i\,) - 2 \,=\, \sum_{i\,=\,1}^{n} 2\,(i-1) \,=\, 2\sum_{k\,=\,1}^{n-1} k \,=\, 2\binom{n}{2} \,=\, n(n-1)\,,$$

the homogeneity claimed for $\mathrm{D}^{\sigma,\, g_R}_n$.

Finally, if g is real analytic so is σ and its covariant derivatives $\nabla^k_{\mathrm{v}}\sigma$ as a function of v, and hence so is $\pi_n\,(\,\|\mathrm{v}\|^2\,\sigma,\,\ldots,\,\|\mathrm{v}\|^2\,\nabla^{n-3}_{\mathrm{v}}\sigma\,)$. ☐

6. Theorem 2. Gauss curvature and the adapted structure on $T^R M$

We show that the functions introduced in the previous sections give a characterization of the existence of the adapted structure in $T^R M$ for a given real analytic and complete Riemannian metric g on M.

For M compact, $T^R M$ with the adapted complex structure is a model for a Grauert tube X^R, which by definition is a complex manifold of dimension n with an exhaustion function $u\colon X^R \to \mathbb{R}$, $u \geq 0$, $\sup u = R$, satisfying, in local holomorphic coordinates $z_1,\,\ldots\,,z_n$:

(i) $\det\left[\partial^2 u^2/\partial z_i\,\partial\bar z_j\right] > 0$ and
(ii) $\det\left[\partial^2 u/\partial z_i \partial\bar z_j\right] = 0$, valid off the closed real analytic manifold $M := u^{-1}(0)$, the fixed-point set of a anti-holomorphic involution of X^R.

By (ii) there is a Kähler metric in X^R with potential u^2 which by restriction gives M a Riemannian metric g; $(M,\,g)$ is called *the center* of the Grauert tube.

Besides the model for X^R in the tangent bundle by Lempert and Szőke where the geometry of the center M is emphasized, there is the one in the cotangent bundle due to V. Guillemin and M. Stenzel [11]) where preeminence is given to the Monge-Ampère function.

It is on that model of Lempert and Szőke where we base our constructions. There X_R is represented as the real manifold

$$T^R M = \{\mathrm{v} \in TM \,|\, g(\mathrm{v},\,\mathrm{v}) \,<\, R^2\} \tag{6.1}$$

endowed with the *adapted complex structure* induced by g [17]. Under this identification, for all $\mathrm{v} \in TM$, $u^2(\mathrm{v}) \,=\, g(\mathrm{v},\mathrm{v}) \,=\, \|\mathrm{v}\|^2 \,=\, 2\,E$, while the map $\mathrm{v} \mapsto -\mathrm{v}$ is the antihololomophic involution. The metric g must be real analytic, by a result of Lempert [16].

As long as g is complete and real analytic, the adapted complex structure is always defined in some neighborhood $\mathcal{U}$ of M in TM. However, when M is not compact such $R > 0$ might not exist, for example if along a geodesic the Gauss curvature is not bounded below thus violating Lempert-Szőke inequality for any $R > 0$. If M is compact such radius of course exist, but its value is severely limited by the geometry of M.

Theorem 2. *Consider a complete real analytic Riemannian metric g on the two-dimensional M. Let $R \in (0,\,\infty]$ be given and $\mathcal{V} \subset TM \setminus M$ any set that maps onto the unit tangent bundle UM by the assignment $\mathrm{v} \mapsto \|\mathrm{v}\|^{-1}\,\mathrm{v}$. Then the adapted complex structure exists up to radius R if and only if for all integer $n \geq 1$ and for all $\mathrm{v} \in \mathcal{V}$,*

$$\mathrm{D}^{\sigma,\, g_R}_n(\mathrm{v}) \,\geq\, 0\,. \tag{6.2}$$

Proof. Since by Proposition 5.3 the functions $\mathrm{D}^{\sigma,\, g_R}_n$ are homogeneous along the fibers of TM, their signs are constant along each ray $\{e^t\,\mathrm{v}\,, t \in \mathbb{R}\} \subset TM\backslash M$ and so, without loss of generality we take $\mathcal{V} \,=\, UM$, the unit tangent bundle.

Consequently, we consider unit speed geodesics in our recollection of the relevant parts of the construction of the adapted complex structure [17].

For a given unit-speed geodesic γ in M consider two Jacobi fields along $\gamma\colon \mathbb{R} \to M$, J_1^γ and J_2^γ, linearly independent and point-wise orthogonal to $\dot\gamma$, and define $h^\gamma\colon \mathbb{R}\setminus \mathbb{S}^\gamma \to \mathbb{R}$ by

$$J_2^\gamma(t) \,=\, h^\gamma(t)\, J_1^\gamma(t),$$

with t arc-length and $\mathbb{S}^\gamma := \{\, t \in \mathbb{R} \mid J_1^\gamma(t) = 0 \,\} \subset \mathbb{R}$.

Consider the vector fields $\tilde J_i^\gamma$, $i = 1, 2$ along the map $\mathbf{P}_\gamma\colon \mathbb{C} \to TM$, where

$$\mathbf{P}_\gamma(\, x + \sqrt{-1}\, y\,) \,=\, y\, \dot\gamma(x)\,, x, y \in \mathbb{R}\,, \tag{6.3}$$

given at $w = x + \sqrt{-1}\, y$, by

$$\tilde J_i^\gamma|_w \,=\, (J_i^\gamma(x))^h_{\mathbf{P}_\gamma(w)} + (\nabla_{\mathbf{P}_\gamma(w)} J_i^\gamma)^v_{\mathbf{P}_\gamma(w)}, \tag{6.4}$$

where, we recall, the *horizontal lift* $(u)^h_{\mathrm{v}}$ and the *vertical lift* $(u)^v_{\mathrm{v}}$ lift of a vector $u \in T_{\pi \mathrm{v}} M$ are the vectors in $T_{\mathrm{v}}(TM)$ defined by

$$\pi_*(u)^h_{\mathrm{v}} \,=\, K(u)^v_{\mathrm{v}} \,=\, u\,,\;\; \pi_*(u)^v_{\mathrm{v}} \,=\, K(u)^h_{\mathrm{v}} \,=\, 0\,,$$

with $K\colon T(TM) \to TM$ the connection map [13], [17].

Then, there is $R > 0$ such the adapted complex structure $\mathbf{J}$ exists on (6.1) if and only if for every unit-speed geodesic, there is a meromorphic extension of h^γ,

$$h^\gamma_{\mathbb{C}}\colon \{\, x + \sqrt{-1}\, y\,;\; x, y \in \mathbb{R},\, |y| \,<\, R\,\} \,\subset\, \mathbb{C} \to \mathbb{C} \cup \{\infty\}$$

with poles only on $\mathbb{S}^\gamma := \{\, t \in \mathbb{R} \mid J_1^\gamma(t) = 0 \,\} \subset \mathbb{R}$ and with positive imaginary part for $y > 0$, such that

$$\tilde J_2^\gamma - \sqrt{-1}\, \mathbf{J}\, \tilde J_2^\gamma = h^\gamma_{\mathbb{C}}(x + \sqrt{-1} y)\, \left(\tilde J_1^\gamma - \sqrt{-1}\, \mathbf{J}\, \tilde J_1^\gamma \right) .$$

On the other hand, along every unit-speed geodesic $\gamma\colon \mathbb{R} \to M$ the map

$$t \mapsto f(t) \star \dot\gamma(t),$$

where $t \mapsto \{\, \dot\gamma(t),\; \star\dot\gamma(t)\,\}$ is an orthonormal frame along γ, is a Jacobi field if and only if

$$f''(t) + \sigma(\gamma(t))\, f(t) \,=\, 0\,; \tag{6.5}$$

hence $h^\gamma = f_2^\gamma / f_1^\gamma$ where f_1^γ and f_2^γ are two independent solutions of (6.5).

Now apply Theorem 1 to every unit-speed γ using the relation (2.14), that establishes the correspondence (2.17), which since we identify σ with $\pi^*\sigma$, becomes

$$\boxed{\vartheta \,\leftrightarrow\, \sigma \circ \gamma}\,.$$

Thus for all integer $k \geq 1$ we have the correspondences

$$\boxed{\vartheta^{(k)} \,\leftrightarrow\, \nabla^k_{\dot\gamma} \sigma}$$

and

$$\boxed{\mathcal{S}_k^{\vartheta, g_R} \,\leftrightarrow\, \mathrm{S}_k^{\sigma, g_R} \circ \dot\gamma}\,.$$

□

In some cases it suffices to check the inequalities in a smaller $\mathcal{V} \subset TM$. For instance we have the following.

Corollary 6.0.1. *Let the Riemannian metric g on M be complete and real analytic, G a subgroup of the isometry group and $\mathcal{Z} \subset TM \setminus M$ any subset of TM whose orbit corresponding to the G-action induced by differentials meets every trajectory of the geodesic flow. The adapted complex structure exists up to radius R if and only if the inequalities (6.2) hold for all integer $n \geq 1$ and for all* $\mathrm{v} \in \mathcal{Z}$.

Proof. By part 2 of Theorem 1 and since the orbits of the geodesic flow on UM correspond by projection to the geodesics of M, in light of Remark 4.1 applied to the context of TM, the real analyticity of g implies that the conditions (6.2) need to be checked at just one point along each orbit of the geodesic flow. As noted earlier due to the re-scaling property of the polynomials π_n shown in Proposition 5.3 the inequalities in Theorem 2 hold along lines through the origin in the fibers of TM, and thus $\mathcal{Z}$ does not have to be contained in the unit tangent bundle UM. □

Remark 6.1. *As an example, consider a real analytic metric of revolution two-sphere $M = S^2$ isometrically embedded in $\mathbb{R}^3$ and with positive Gauss curvature. Then Corollary 6.0.1 with $G = S^1$ applies, with $\mathcal{Z} = U_x M$ for x any point in the equator of M.*

Remark 6.2. *On $TM \setminus M$, there is a canonical frame field $z \mapsto \{X_1, X_2, V_3, V_4\}$ defined by the Levi-Civita connection and the endomorphism $S\colon TM \to TM$ given by a positive rotation of $\pi/2$ with respect to g.*

1. *X_1 is the geodesic spray, the geodesic flow given by $(\mathrm{v},\, t) \mapsto \phi_t(\mathrm{v}) := \dot\gamma_{\mathrm{v}}(t)$ with γ_{v} the constant speed geodesic with $\dot\gamma_{\mathrm{v}}(0) = \mathrm{v}$;*
2. *V_1 is the radial vector field, $(\mathrm{v},\, t) \mapsto e^t\, \mathrm{v}$;*
3. *X_2 has as flow $(\mathrm{v}, t) \mapsto \mathrm{j}\, \phi_t\, \mathrm{j}^{-1}(\mathrm{v}) = \mathrm{j}\, \dot\gamma_{\mathrm{j}^{-1}\mathrm{v}}(t)$, where and $\gamma_{\mathrm{j}^{-1}\mathrm{v}}$ is the unit speed geodesic with $\dot\gamma_{\mathrm{j}^{-1}\mathrm{v}}(0) = -\mathrm{j}\mathrm{v}$;*
4. *V_2 is the angular field with flow $(\mathrm{v},\, t) \mapsto \cos(t)\, \mathrm{v} + \sin(t)\, \mathrm{j}\mathrm{v}$.*

In terms of the horizontal and vertical lifts we have

$$X_1|_{\mathrm{v}} = (\mathrm{v})^h_{\mathrm{v}}, \quad V_1|_{\mathrm{v}} = (\mathrm{v})^v_{\mathrm{v}}, \quad X_2|_{\mathrm{v}} = (\mathrm{j}\mathrm{v})^h_{\mathrm{v}}, \quad V_2|_{\mathrm{v}} = (\mathrm{j}\mathrm{v})^v_{\mathrm{v}},$$

and the relation with the notation as above by the equation for normal Jacobi fields

$$(\phi_t)_* \begin{bmatrix} X_2 \\ V_2 \end{bmatrix}_{\mathrm{v}} = \begin{bmatrix} f_1 & f_1' \\ f_2 & f_2' \end{bmatrix}_t \begin{bmatrix} X_2 \\ V_2 \end{bmatrix}_{\phi_t \mathrm{v}},$$

interpreted along unit-speed geodesics $\gamma\colon \mathbb{R} \to M$ where the assignments

$$s \mapsto J_i(t) = f_i(t)\, X_2|_{\phi_t \dot\gamma(0)}, i = 1,\, 2$$

define linearly independent Jacobi fields along γ. Now, using these Jacobi fields one proves that

$$[X_1, X_2]_{\mathrm{v}} = \sigma\, \|\mathrm{v}\|^2\ V_2, \tag{6.6}$$

which of course defines the Gauss curvature σ as the obstruction for the integrability of the horizontal distribution determined by the connection and spanned by X_1 and X_2. The equation (6.6) is the reason for the introduction of the factor $\|\mathrm{v}\|^2$ in (5.14), so that the Schwarzians in TM re-scale correctly along the fibers. It also indicates that the Schwarzians in TM can be defined equivalently in terms to derivatives of σ along the geodesic flow.

In fact the version in TM of the formula (2.10) is

$$\mathrm{S}^\sigma_{n+1}(\mathrm{v}) = \mathrm{d}\, \mathrm{S}^\sigma_n\, (X_1)\, (\mathrm{v}) + \sigma \sum_{k=1}^{n-1} \binom{n}{k}\, \mathrm{S}^\sigma_k(\mathrm{v})\, \mathrm{S}^\vartheta_{n-k}(\mathrm{v})\,. \tag{6.7}$$

7. On the rank of the infinite Schwarzian matrix

We include some comments on the rank of the infinite Hankel matrix of g_R-Schwarzians and its relation to the adapted complex structure. A companion infinite Hankel matrix is introduced in Definition 8.2 and applied to closed geodesics.

Definition 7.1. *Put, for $\mathrm{v} \in TM$,*

$$\mathrm{R}^{\sigma,\, g_R}(\mathrm{v}) := \mathrm{Rank} \left[\frac{\mathrm{S}^{\sigma,\, g_R}_{i+j-1}(\mathrm{v})}{(i+j-1)!} \right]_{i,\, j=1}^{\infty}, \tag{7.1}$$

and consider the function $\mathrm{R}^{\sigma,\, g_R}\colon TM \to \mathbb{N} \cup \{\infty\}$ defined by letting $\mathrm{v} \in TM$ vary in (8.1).

We will show by means of Proposition 7.1 and Proposition 7.3 that this function is constant along the leaves of the foliation of $TM \setminus M$,

$$TM \setminus M = \bigcup_{\gamma \text{ geodesic of M}} \mathbf{P}_\gamma\, (\mathrm{C} \setminus \mathbb{R})\,,$$

determined by the images of the maps (6.3).

Note that for all $\mathrm{v} \in M \subset TM$ we always have $\mathrm{R}^{\sigma,\, g_R}(\mathrm{v}) = 1$, by Remark 5.4.

Proposition 7.1. *The function $\mathrm{R}^{\sigma,\, g_R}$ is constant along the lines in the fibers of TM, i.e., for all $\mathrm{v} \in TM$ and for all $t \in \mathbb{R}$*

$$\mathrm{R}^{\sigma,\, g_R}(e^t\, \mathrm{v}) = \mathrm{R}^{\sigma,\, g_R}(\mathrm{v})\,. \tag{7.2}$$

Proof. Put, for all integer $n \geq 1$,

$$\mathrm{C}_n(\mathrm{v}) \;=\; \frac{\mathrm{S}_n^{\sigma,\, g_R}(\mathrm{v})}{n!}\,.$$

The result claimed is a consequence of the homogeneity property

$$\mathrm{C}_n(\lambda\,\mathrm{v}) \;=\; \lambda^{n-1}\,\mathrm{C}_n(\mathrm{v}) \tag{7.3}$$

proved in Proposition 5.3 and the special symmetry of an infinite Hankel matrix .

In fact, by Theorem 7 in [10], the infinite Hankel matrix

$$[\mathrm{C}_{i+j-1}(\mathrm{v})]_{i,j=1}^{\infty} \tag{7.4}$$

has finite rank N if and only if N is the smallest natural number such that the $(1+N)^{\text{th}}$ infinite column of (7.4) is a linear combination of the $1^{\text{st}}, \cdots N^{\text{th}}$ infinite columns of this infinite matrix, i.e., if and only N is the smallest natural number so that there are numbers $a_1(\mathrm{v}), \ldots a_N(\mathrm{v}) \in \mathbb{C}$ such that

$$\begin{bmatrix} \mathrm{C}_1(\mathrm{v}) & C_2(\mathrm{v}) & \cdots & \mathrm{C}_N(\mathrm{v}) \\ \mathrm{C}_2(\mathrm{v}) & \mathrm{C}_3(\mathrm{v}) & \cdots & \mathrm{C}_{N+1}(\mathrm{v}) \\ \vdots & \vdots & & \vdots \end{bmatrix} \begin{bmatrix} a_1(\mathrm{v}) \\ \vdots \\ a_N(\mathrm{v}) \end{bmatrix} = \begin{bmatrix} \mathrm{C}_{N+1}(\mathrm{v}) \\ \mathrm{C}_{N+2}(\mathrm{v}) \\ \vdots \end{bmatrix}, \tag{7.5}$$

which is equivalent to the validity of the sequence of equalities

$$\mathrm{C}_{N+n+1}(\mathrm{v}) = \sum_{i=1}^{N} a_i(\mathrm{v})\,\mathrm{C}_{i+n}(\mathrm{v}) \tag{7.6}$$

for all integer $n \geq 0$. But from (7.3), for any $\lambda \neq 0$, the equalities in the sequence (7.6) hold for the given $a_1(\mathrm{v}), \cdots, a_N(\mathrm{v})$ if and only if the equalities in the sequence

$$\mathrm{C}_{N+n+1}(\lambda\,\mathrm{v}) = \sum_{i=1}^{N} a_i(\lambda\mathrm{v})\,\mathrm{C}_{i+n}(\mathrm{v}),$$

where $n \geq 0$, hold for the coefficients $a_i(\lambda\mathrm{v})$ given for $i = 1, \cdots, N$ by

$$a_i(\lambda\mathrm{v}) \;:=\; \lambda^{N-i+1}\,a_i(\mathrm{v})\,.$$

□

We will use the following facts taken from [10].

Proposition 7.2. *Let be given an infinite Hankel matrix*

$$[\mathrm{C}_{i+j-1}]_{i,j=1}^{\infty}\;. \tag{7.7}$$

(i) *(Corollary of Theorem 7 in* [10]*) If the matrix (7.7) has finite rank N, then*

$$\det\,[\mathrm{C}_{i+j-1}]_{i,j=1}^{N} \neq 0\,. \tag{7.8}$$

(ii) *(Theorem 8 in* [10]*) The matrix (7.7) has finite rank N if and only if $\sum_{n=1}^{\infty} \mathrm{C}_n\,u^{-n}$ is rational in u of degree N.*

Proposition 7.3. *The function $\mathrm{R}^{\sigma,\, g_R}$ is constant along the trajectories of the geodesic flow on TM, that is, for all $x \in \mathbb{R}$,*

$$\mathrm{R}^{\sigma,\, g_R}(\phi_x\mathrm{v}) \;=\; \mathrm{R}^{\sigma,\, g_R}(\mathrm{v})\,. \tag{7.9}$$

Proof. We already noted that $\mathrm{R}^{\sigma,\, g_R}(\mathrm{v}) = 1$ for any $\mathrm{v} \in M \subset TM$. Moreover, in light of Proposition 7.1, it suffices to consider the flow on the unit tangent bundle $UM \subset TM$. So let $\mathrm{v} \in TM$ with $\|\mathrm{v}\| = 1$ and take the unit-speed geodesic $\gamma\colon \mathrm{R} \mapsto M$ given by $\gamma(t) = \pi\,(\phi_t\mathrm{v})$.

With notation as in the proofs of Theorem 1 and Theorem 2, for any $x \in \mathrm{R}$, take the quotient h_x of the two fundamental solutions at x of equation in (6.5), and consider the function of s

$$(h_x \circ T_x \circ g_R)\,(s) \;=\; h_x\,(x + g_R(s)) \tag{7.10}$$

which is defined for $s \in (\,-\epsilon,\,\epsilon) \subset \mathrm{R}$ for some $\epsilon > 0$. Since for all integer $n \geq 0$,

$$(h_x \circ T_x \circ g_R)^{(n)}(0) \;=\; \mathrm{S}_n^{\sigma,\, g_R}(\phi_x\mathrm{v}),$$

then

$$\mathrm{R}^{\sigma,g_R}(\phi_x \mathrm{v}) = \mathrm{Rank}\left[\frac{(h_x \circ T_x \circ g_R)^{(i+j-1)}(0)}{(i+j-1)!}\right]_{i,j=1}^{\infty}.$$

Now, by Proposition 7.2 and the transformation of degree 1, $u \mapsto t^{-1}$, it follows that the rank of the infinite matrix in the right-hanside is $N < \infty$ if and only if

$$(h_x \circ T_x \circ g_R)(s)$$

is a rational function in s of degree N. Such value of N is independent of x since for any x_1 and x_2 in $\mathbb{R}$ the quotients h_{x_1} and h_{x_2} are related by a (real) Moëbius transformation, which is rational of degree 1. ☐

As a consequence the following aspect of Theorem 7 in [10] appears in the context of the Schwarzian functions.

Corollary 7.3.1. *With notation as in (2.15), for* $\mathrm{v} \in TM$ *it holds*

$$\mathrm{R}^{\sigma,\, g_R}(\mathrm{v}) = N < \infty, \tag{7.11}$$

if N *is the smallest natural number so that for each* $n \geq 1$ *the function of* t

$$\mathrm{S}^{\sigma,\, g_R}_{2N+n}(\phi_t \mathrm{v})$$

can be written, for all $t \in \mathbb{R}$, *in terms of the set of functions of* t

$$\left\{\mathrm{S}^{\sigma,\, g_R}_{1}(\phi_t \mathrm{v}),\, \ldots,\, \mathrm{S}^{\sigma,\, g_R}_{2\,N}(\phi_t \mathrm{v})\right\}$$

as

$$\mathrm{S}^{\sigma,\, g_R}_{2N+n}(\phi_t \mathrm{v}) \;=\; \frac{p_{n,\,N}\left(\mathrm{S}^{\sigma,\, g_R}_{1}(\phi_t \mathrm{v}), \cdots, \mathrm{S}^{\sigma,\, g_R}_{2N}(\phi_t \mathrm{v})\right)}{\left(\mathrm{D}^{\sigma,\, g_R}_{N}(\phi_t \mathrm{v})\right)^{n+1}}, \tag{7.12}$$

where $p_{n,N} = p_{n,\,N}(X_1, \cdots, X_{2N})$ *are universal polynomials with coefficients dependent on* t. *(The statement includes* $\mathrm{S}^{\sigma,\, g_R}_{1}(\phi_t \mathrm{v}) \equiv 1$, *for convenience).*

Proof. Set, for all integer $n \geq 1$ and all $t \in \mathbb{R}$,

$$\mathrm{C}_n(t) \;=\; \frac{\mathrm{S}^{\sigma,g_R}_{n}(\phi_t \mathrm{v})}{n!},$$

and

$$\mathrm{D}_N(t) \;=\; \mathrm{D}^{\sigma,\, g_R}_{N}(\phi_t \mathrm{v}).$$

The fact, proven in Proposition 7.3, that the rank of the infinite matrix $[\mathrm{C}_{i+j-1}(t)]_{i,\,j=1}^{\infty}$ is independent of t, together with Proposition 7.2 and the sequence of equalities (7.6), means that the noted rank is $N < \infty$ if and only if such is the smallest natural number with the property that there are functions $a_1(t), \cdots, a_N(t)$ defined for all $t \in \mathbb{R}$ so that for every integer $n \geq 0$,

$$[\mathrm{C}_{i+j+n-1}(t)]_{i,\,j=1}^{N} \begin{bmatrix} a_1(t) \\ \vdots \\ a_N(t) \end{bmatrix} = \begin{bmatrix} \mathrm{C}_{N+1+n}(t) \\ \vdots \\ \mathrm{C}_{2N+n}(t) \end{bmatrix}. \tag{7.13}$$

From the equality for $n = 0$, namely

$$[\mathrm{C}_{i+j-1}(t)]_{i,\,j=1}^{N} \begin{bmatrix} a_1(t) \\ \vdots \\ a_N(t) \end{bmatrix} = \begin{bmatrix} \mathrm{C}_{N+1}(t) \\ \vdots \\ \mathrm{C}_{2N}(t) \end{bmatrix}, \tag{7.14}$$

and induction, the equalities above are equivalent to

$$\begin{bmatrix} \mathrm{C}_{N+1+n}(t) \\ \vdots \\ \mathrm{C}_{2\,N+n}(t) \end{bmatrix} = [\mathrm{C}_{i+j-1}(t)]_{i,\,j=1}^{N} \underbrace{\begin{bmatrix} 0 & 0 & \ldots & 0 & a_1(t) \\ 1 & 0 & \ldots & 0 & a_2(t) \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \ldots & 0 & 1 & a_N(t) \end{bmatrix}^{n} \begin{bmatrix} a_1(t) \\ \vdots \\ a_N(t) \end{bmatrix}}_{(*)}, \tag{7.15}$$

for all integer $n \geq 0$, exhibiting the functions $\mathrm{C}_{2N+n}(t)$ for every integer $n \geq 0$ as rational in $\mathrm{C}_2(t), \cdots, \mathrm{C}_{2N}(t)$. Now, since $\mathrm{D}_N(t) \neq 0$ for all $t \in \mathbb{R}$ by Proposition 7.2 and Proposition 7.3, that the particular form of such dependence is as claimed follows by writing (*) in (7.15) as

$$(\mathrm{D}_N(t))^{-n-1} \begin{bmatrix} 0 & 0 & \dots & 0 & p_1 \\ \mathrm{D}_N(t) & 0 & \dots & 0 & p_2 \\ 0 & \mathrm{D}_N(t) & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \dots & 0 & \mathrm{D}_N(t) & p_N \end{bmatrix}^n \begin{bmatrix} p_1 \\ \vdots \\ p_N \end{bmatrix},$$

where, from Cramer's rule in (7.14), for $i = 1, \dots, N$,

$$\mathrm{p}_{i,N}(\mathrm{C}_1(t) \cdots, \mathrm{C}_{2N}(t)) = a_i(t)\,\mathrm{D}_N(t)\,, \tag{7.16}$$

with $\mathrm{p}_{i,N}$ polynomials in the $\mathrm{C}_k(t)$ for k as indicated. □

We introduce the following generating functions.

Definition 7.2. *Let $\mathrm{G}^{\sigma, g_R}\colon TM \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by*

$$\mathrm{G}^{\sigma, g_R}(\mathrm{v}, x, s) = \sum_{n=1}^{\infty} \mathrm{S}_n^{\sigma, g_R}(\phi_x \mathrm{v}) \frac{s^n}{n!}\,. \tag{7.17}$$

Here g_R is given as before, according to Definition 3.2. In particular for $R = \infty$ the above reduces to

$$\mathrm{G}^{\sigma}(\mathrm{v}, x, s) := \sum_{n=1}^{\infty} \mathrm{S}_n^{\sigma}(\phi_x \mathrm{v}) \frac{s^n}{n!}\,. \tag{7.18}$$

Then Proposition 7.2 and Proposition 7.3 imply the following.

Corollary 7.3.2. *Let $R \in (0, \infty]$ and $\mathrm{v} \in UM$. Then $\mathrm{R}^{\sigma, g_R}(\mathrm{v}) = N < \infty$ if and only if for all $x \in \mathbb{R}$ the generating function $\mathrm{G}^{\sigma, g_R}(\mathrm{v}, x, s)$ is rational in s of degree N.*

Corollary 7.3.3. *Let the Riemannian metric g on M be real analytic, and to be specific, consider $g_R(s) = -\dfrac{R}{\pi} \ln\left(1 - \dfrac{\pi s}{R}\right)$. Given $R \in (0, \infty]$ and $\mathrm{v} \in UM$, $\mathrm{R}^{\sigma, g_R}(\mathrm{v}) = N < \infty$ if and only if*

$$\mathrm{G}^{\sigma}(\mathrm{v}, x, s) = \mathcal{R}(\mathrm{v}, x, u(s)), \tag{7.19}$$

with $\mathcal{R}$ is rational in u of degree N and

$$u(s) = \frac{R}{\pi}\,(1 - e^{\frac{\pi}{R} s})\,, \tag{7.20}$$

interpreted as $u = s$ when $R = \infty$.

Moreover if the adapted complex structure is defined on the entire TM, for every $x \in \mathbb{R}$ the poles and zeros of $\mathcal{R}(\mathrm{v}, x, u)$, as a rational function of u, are all simple and real, the poles with negative residues.

Proof. With notation as in Proposition 7.3, taking the unit-speed geodesic $\gamma\colon \mathrm{R} \mapsto M$ with $\dot\gamma(0) = \mathrm{v}$, for each $x \in \mathbb{R}$ we have the equality of power series

$$\begin{aligned} \mathrm{G}^{\sigma, g_R}(\mathrm{v}, x, t) &= \sum_{n=1}^{\infty} \mathrm{S}_n^{\sigma, g_R}(\phi_x \mathrm{v}) \frac{t^n}{n!} \\ &= \sum_{n=1}^{\infty} (h_x \circ T_x \circ g_R)^{(n)}(0) \frac{t^n}{n!}\,. \end{aligned} \tag{7.21}$$

By the real analyticity of g and given the choice of g_R the power series converges near $t = 0$, thus

$$\mathrm{G}^{\sigma, g_R}(\mathrm{v}, x, t) = h_x\left(x - \frac{R}{\pi} \ln(1 - \frac{\pi t}{R})\right)\,. \tag{7.22}$$

By Corollary 7.3.2 the left hand-side is rational in t of degree N, say

$$\mathrm{G}^{\sigma, g_R}(\mathrm{v}, x, t) = \mathcal{R}^{\sigma, g_R}(\mathrm{v}(\mathrm{v}, x, t), \tag{7.23}$$

if and only if $\mathrm{R}^{\sigma,\,g_R}(\phi_x \mathrm{v}) = N < \infty$, this rank equal to $\mathrm{R}^{\sigma,\,g_R}(\mathrm{v})$ by Proposition 7.3.

So, the first part of the Corollary follows from (7.24), (7.23) and real analyticity by observing that

$$
\begin{aligned}
\mathcal{R}^{\sigma,\,g_R}(\mathrm{v}(x,\,\mathrm{v},\,\frac{R}{\pi}(1-e^{-\frac{\pi}{R}s})) &= h_x(x+s) \\
&= \sum_{n=1}^{\infty} h_x^{(n)}(x)\frac{s^n}{n!} \\
&= \sum_{n=1}^{\infty} \mathrm{S}_n^{\sigma}(\phi_x \mathrm{v})\,\frac{s^n}{n!} \\
&= \mathrm{G}^{\sigma}(\mathrm{v},\,x,\,s)\,.
\end{aligned}
\tag{7.24}
$$

In addition, if the adapted structure is defined on the whole TM, for each $x \in \mathbb{R}$ the function $h_x(x+t)$ on the right-hand side of (7.24) is a Pick function in t. Thus, in light of the fact that $g_R(t) \in \mathbb{R}$ precisely when $t \in \mathbb{R}$ the conditions imposed on $\mathcal{R}^{\sigma,\,g_R}(\mathrm{v},\,x,\,u)$ as a rational function in u follow. □

8. Schwarzians with purely imaginary radius and closed geodesics

To complete our remarks on higher order Schwarzians we relate formally the periodicity of the geodesic flow along a trajectory with the the finiteness of the rank of an infinite Hankel matrix now constructed with complex valued higher order Schwarzians defined by considering complex values for the radius R as indicated in Remark 3.6 for the Schwarzians in $\mathbb{R}$.

Definition 8.1. *For any $0 \neq \lambda \in \mathbb{R}$ and integer $n \geq 0$ put*

$$
\boxed{\mathrm{S}_n^{\sigma,\,g_{\sqrt{-1}\lambda}} := \mathrm{S}_n^{\sigma,\,g_R}\big|_{R=\sqrt{-1}\lambda}}\,.
$$

A companion to Definition 7.1 is the following.

Definition 8.2. *Given $\mathrm{v} \in TM$ put*

$$
\mathrm{R}^{\sigma,\,g_{\sqrt{-1}\lambda}}(\mathrm{v}) := \mathrm{Rank}\left[\frac{\mathrm{S}_{i+j-1}^{\sigma,\,g_{\sqrt{-1}\lambda}}(\mathrm{v})}{(i+j-1)!}\right]_{i,\,j=1}^{\infty}, \tag{8.1}
$$

and consider the function $\mathrm{R}_{\mathbb{C}}^{\sigma,\,g_R}\colon TM \to \mathbb{N} \cup \{\infty\}$ defined by letting $\mathrm{v} \in TM$ vary in (8.1).

Proposition 8.1. *(Periodicity of Jacobi fields and the adapted complex structure). Let M be a two-dimensional real analytic manifold with a real analytic complete Riemannian metric g with the adapted complex structure defined on the entire TM. Let $\gamma\colon \mathbb{R} \to M$ be any unit speed closed geodesic of M.*

Take $g_R = \dfrac{R}{\pi}\ln(1-\dfrac{\pi}{R}z)$.

Then if the normal Jacobi fields[3] *along γ are periodic with period T then there is an integer $N \geq 1$ so that for all $t \in \mathbb{R}$,*

$$
\mathrm{R}_{\mathbb{C}}^{\sigma,\,g_{\sqrt{-1}\frac{T}{2}}}(\dot{\gamma}(\mathrm{t})) = N\,. \tag{8.2}
$$

Proof. Let γ be a unit speed geodesic parameterized of M. Let $x \in \mathbb{R}$ be fixed, and set $h_x(t) = f_{x,\,2}(t)/f_{x,\,1}(t)$, with $f_{x,\,1}$ and $f_{x,\,2}$ the pair of fundamental solutions of $f''(t) + \sigma(\gamma(t))f(t) = 0$ at x in the sense of (3.9), each function identified with a normal Jacobi field as in the proof of Theorem 2.

If all the Jacobi fields normal to γ are periodic with period T then

$$
h_x(t) = h_x(t+T) \tag{8.3}
$$

for all $t \in \mathbb{R} \setminus f_{x,\,1}^{-1}(0)$.

[3]Normal means point-wise orthogonal to $\dot{\gamma}$

Since the adapted complex structure is defined on the entire TM, $h_x(z)$ is a Pick function in z, hence representable, according to Fatou's formula, for $z \in \mathbb{C} \setminus \mathbb{R}$, by

$$h_x(z) \,=\, \alpha\, z \,+\, \beta \,+ \int_{-\infty}^{\infty} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) d\mu(t), \tag{8.4}$$

with constants $\alpha = \alpha_x$, $\beta = \beta_x \in \mathbb{R}$, $\alpha_x \geq 0$, and with $d\mu(t) = d\mu_x(t)$ a non-negative Borel measure with

$$\int_{-\infty}^{\infty} \frac{d\mu(t)}{1+t^2} \,<\, \infty\,.$$

The measure $d\mu(t)$ is determined by a weak limit,

$$\int_a^b d\mu(t) \,=\, \lim_{y \mapsto 0+} \frac{1}{\pi} \int_a^b \Im h_x(t \,+\, \sqrt{-1}y)\, dt\,.$$

Thus, in our particular case, $h_x(z)$ is meromorphic is z, with poles that are all real and simple, namely the zeros of $f_{x,1}(t)$, with negative residues.

In addition, from (8.3), we have $\alpha = 0$ in (8.4), and moreover, for some integer

$$N \,=\, N(x,T) \,\geq 1, \tag{8.5}$$

the polar set, $f_{x,1}^{-1}(0)$, is of the form

$$\{a_1 + k\,T,\, \cdots\,,\, a_N + k\,T\,,\, k \in \mathbb{Z}\}$$

where all the $a_i = a_i(x)$ are in $\mathbb{R}$ and satisfy

$$x - \frac{T}{2} < a_1 < \cdots < a_N \leq x + \frac{T}{2}\,. \tag{8.6}$$

The support of $d\mu(t)$ consists of a mass at each point in $\{a_i +\, k\,T,\, k \in \mathbb{Z},\, i = 1, \cdots, N\}$ with weight

$$r_i \,=\, r_i(x) \,=\, \lim_{t \mapsto a_i} (t - a_i)\, h_x(t) \,>\, 0\,.$$

Thus, using Herglotz formula,

$$\begin{aligned} h_x(z) \quad &= \quad \sum_{i=1}^{N} \frac{r_i}{z - a_i} + r_i \sum_{k=1}^{\infty} \left(\frac{1}{z - a_i - k\,T} + \frac{1}{z - a_i + k\,T} \right) \\ &= \quad \frac{\pi}{T} \sum_{i=1}^{N} \frac{r_i}{\tan \frac{\pi}{T}(z - a_i)}\,. \end{aligned} \tag{8.7}$$

Now, observe that

$$F_{x,\lambda}(z) := h_x \left(x \,-\, \frac{\sqrt{-1}\lambda}{\pi} \ln\left(1 - \frac{\pi}{\sqrt{-1}\lambda} z\right)\right) \tag{8.8}$$

is rational in z if we take

$$\lambda \,=\, n\frac{T}{2}\,,$$

for arbitrary integer $n \neq 0$. To check this explicitly, for $n = 1$, consider the contributions to (8.8) of each of the terms in (8.7). Use the identity

$$\tan\,(a - b) \,=\, \frac{\tan a \,-\, \tan b}{1 + \tan a \tan b}$$

with $a = x - a_i$, and the identity

$$\tan \left(\frac{\sqrt{-1}}{2}\, \ln\,(1 - \frac{2\,\pi}{\sqrt{-1}T} z) \right) \quad = \quad \frac{\sqrt{-1}z}{z - \sqrt{-1}\,\frac{T}{\pi}}\,, \tag{8.9}$$

to conclude that, for each $1 \leq i \leq N$,

$$\frac{1}{\tan \left(\frac{\pi}{T}(x - a_i) \,-\, \frac{\sqrt{-1}}{2} \ln\,(1 - \frac{2\,\pi}{\sqrt{-1}T} z) \right)} \tag{8.10}$$

simplifies to the Moëbius transformation in z

$$\frac{\sqrt{-1}\; z + \frac{T}{\pi} A_i}{z - \sqrt{-1}\,(1 - A_i)\,\frac{T}{\pi}}, \tag{8.11}$$

with

$$A_i \;=\; A_i(x,T) \;=\; e^{\frac{\pi}{T}(x-a_i)\sqrt{-1}}\;\cos\frac{\pi}{T}(x-a_i)\,. \tag{8.12}$$

Hence, $F_{x,\frac{T}{2}}(z)$ is rational in z, of degree depending on T, but not on x, since $F_{x_1,\frac{T}{2}}(z)$ is related to $F_{x_2,\frac{T}{2}}(z)$ by a Moëbius trasformation. Moreover, by inspection of the denominators of (8.11), since by (8.6) we have that for all $1 \le i,j \le N$

$$A_i \ne A_j, \quad i \ne j\,,$$

it follows that the degree of the rational function $F_{x,\frac{T}{2}}(z)$ in z is the number N, the possible non-constancy of which, indicated in (8.5), can be now fully described by $N = N(T)$.

So, we have

$$\boxed{F_{x,\frac{T}{2}}(z) = \frac{\pi}{T}\sum_{i=1}^{N(T)} r_i(x)\;\frac{\sqrt{-1}\; z + \frac{T}{\pi}A_i(x,T)}{z - \sqrt{-1}\,(1 - A_i(x,T))\,\frac{T}{\pi}}}\,.$$

Now, the function $F_{x,\lambda}(z)$ given by (8.8) is holomorphic in a neighborhood of $z = 0 \in \mathbb{C}$, and from the definitions we have

$$F^{(n)}_{x,\lambda}(0) \;=\; \mathrm{S}_n^{\sigma,\,g_{\sqrt{-1}\lambda}}(\dot\gamma(x))\,,$$

and for $\lambda = \frac{T}{2}$,

$$\mathrm{R}^{\sigma,\,g_{\sqrt{-1}\frac{T}{2}}}(\dot\gamma(x)) \;=\; \mathrm{Rank}\left[\frac{\mathrm{S}_{i+j-1}^{\sigma,\,g_{\sqrt{-1}\frac{T}{2}}}(\dot\gamma(x))}{(i+j-1)!}\right]_{i,j=1}^{\infty} \;=\; \mathrm{Rank}\left[\frac{F^{(i+j-1)}_{x,\frac{T}{2}}(0)}{(i+j-1)!}\right]_{i,j=1}^{\infty}.$$

Thus, by Proposition 7.2 and Proposition 7.3 used exactly as in the previous section,

$$\mathrm{R}^{\sigma,\,g_{\sqrt{-1}\frac{L}{2}}}(\dot\gamma(t)) = \mathrm{R}^{\sigma,\,g_{\sqrt{-1}\frac{T}{2}}}(\dot\gamma(x)) = N$$

for all $t \in \mathbb{R}$. $\square$

Corollary 8.1.1. *Let M be a two-dimensional closed real analytic Riemannian manifold with all geodesics closed of length L and with adapted complex structure on the entire TM. Then for any unit speed geodesic γ and for all* $\mathrm{t} \in \mathbb{R}$

$$\mathrm{Rank}\left[\frac{\mathrm{S}_{i+j-1}^{\sigma,\,g_{\sqrt{-1}\frac{L}{2}}}(\dot\gamma(t))}{(i+j-1)!}\right]_{i,j=1}^{\infty} \;\le\; \frac{L}{\pi}\max_{x\in M}\sqrt{\sigma}\,. \tag{8.13}$$

Proof. By the standard Sturm comparison Theorem, the distance d between zeros of a normal Jacobi field along any unit speed geodesic γ satisfies

$$d \ge \frac{\pi}{\sup_{t\in\mathbb{R}}\sqrt{\sigma(\gamma(t))}}\,.$$

In the present situation, the normal Jacobi fields along any unit speed geodesic are all periodic, and so will be their quotients, with period L. Thus, with the notation as in the proof of Proposition 8.1 we must have $N \le L/d$, and (8.13) follows. $\square$

Remark 8.1. *Thus in a situation as in the Corollary 8.1.1 we have the following canonical maps. Let*

$$N = \lfloor \frac{L}{\pi}\max_{x\in M}\sqrt{\sigma}\rfloor\,.$$

For any $k \ge 1$ the map $TM \to \mathbb{C}^{2(N+k)-1}$

$$\mathrm{v} \in TM \to \big(Z_1(\mathrm{v}),\;\cdots\;,\;Z_{2(N+k)-1}(\mathrm{v})\big) \in \mathbb{C}^{2(N+k)-1}$$

with

$$Z_i(\mathrm{v}) := \frac{1}{k!}\mathrm{S}_k^{\sigma,\, g_{\sqrt{-1}\frac{L}{2}}}(\mathrm{v})\,, \quad i = 1, \cdots, 2(N+k)-1\,, \tag{8.14}$$

sends TM *to*

$$\{(Z_1, \cdots, Z_{2(N+k)-1}) \in \mathbb{C}^{2(N+k)-1} |\, \det\left[Z_{i+j-1}\right]_{i,\, j=1}^{N+k} = 0\}\,.$$

The map is determined by its restriction to the unit tangent bundle UM since by the homogeneity property of the Schwarzians as defined,

$$Z_i(a\mathrm{v}) = a^{i-1} Z_i(\mathrm{v})\,. \tag{8.15}$$

Remark 8.2. *In case of M is the 2-sphere with constant curvature 1 the period T in (8.3) can be taken to be π rather than 2π, obviously for all unit-speed geodesics.*

It follows that (8.13) is also valid if we use instead of the value $L = 2\pi$ we use $L = \pi$. This of course is consistent with that (8.13) applies to projective space as well. With this value of L the rank of the infinite matrix of the indicated Schwarzians is 1, corresponding to the fact that the entries, for $\|\mathrm{v}\| = 1$, are the McLaurin coefficients of the right-hand side of (8.9).

In this simple case the functions (8.14) are constant along the unit tangent bundle and satisfy (8.15). Thus, since $Z_1 \equiv 1$, the image of TM by the map determined by the Z_i above, for any integer $k > 1$, is just one point in a weighted projective space.

9. Some computations

9.1. A few Schwarzians $\mathcal{S}_n^\vartheta$ in $\mathbb{R}$ in terms of curvature ϑ.

We get the first few Schwarzians in $\mathbb{R}$, (besides $\mathcal{S}_1^\vartheta = 1$, and $\mathcal{S}_2^\vartheta = 0$) using the method in Proposition 5.2, that we will need for some of the examples that follow.

We consider $x = 0$ and $f_1 = f_1(t)$ and $f_2 = f_2(t)$ as in (5.4). Differentiate twice the identity $f_1\, h = f_2$ to get, using (5.3),

$$0 = 2\, f_1'\, h' + f_1\, h''. \tag{9.1}$$

Differentiate (9.1), use (3.1) and get

$$0 \;=\; \left(-2\,\vartheta h' + h'''\right)\, f_1 + 3\, f_1'\, h''\,; \tag{9.2}$$

evaluation at $x = 0$ and (5.4) gives $h'''(0) = 2\,\vartheta(0) \Rightarrow \boxed{\mathcal{S}_3^\vartheta = 2\,\vartheta}$.

Differentiate (9.2); by (5.3),

$$0 \;=\; \left(-2\,\vartheta'\, h' - 5\,\vartheta\, h'' + h^{(4)}\right)\, f_1 + \left(-2\,\vartheta\, h' + 4h'''\right)\, f_1'; \tag{9.3}$$

set $x = 0$; by (5.4) $h^{(4)}(0) = 2\,\vartheta'(0) \Rightarrow \boxed{\mathcal{S}_4^\vartheta = 2\,\vartheta'}$.

Differentiate (9.3), use (5.3) and set $x = 0$; by (5.4) and $h'''(0) = 2\,\vartheta(0)$ obtained earlier, get $h^{(5)}(0) = 2\,\vartheta''(0) + 16\,\vartheta^2(0) \Rightarrow \boxed{\mathcal{S}_5^\vartheta = 2\,\vartheta'' + 16\,\vartheta^2}$.

In a similar fashion we get

$$\boxed{\mathcal{S}_6^\vartheta = 2\,\vartheta''' + 52\,\vartheta\,\vartheta'}$$

and

$$\boxed{\mathcal{S}_7^\vartheta = 2\,\vartheta^{(4)} + 76\,\vartheta''\,\vartheta + 52\,\left(\vartheta'\right)^2 + 272\,\vartheta^3}$$

.

9.2. The Schwarzians $\mathcal{S}_n^{\vartheta,\, g_R}$ for $g_R(z) = -\frac{R}{\pi}\ln(1 - \frac{\pi z}{R})$.

We know make a choice of g_R and compute $\mathcal{S}_n^{\vartheta,\, g_R}$ in terms of the $\mathcal{S}_i^\vartheta$.

Proposition 9.1. *For and given $R \in (0, \infty]$ consider*

$$\boxed{g_R(z) = -\frac{R}{\pi}\ln(1 - \frac{\pi\, z}{R})}, \tag{9.4}$$

defined in $\mathbb{C} \setminus [\,\frac{R}{\pi},\infty\,)$ *with* $-\pi < \arg(1-\frac{\pi z}{R}) < \pi$. *For all integer* $n \geq 1$ *the* g_R*-Schwarzians in* $\mathbb{R}$ *are*

$$\boxed{\mathcal{S}_n^{\vartheta,\,g_R} \;=\; \sum_{k=1}^{n} \left[\!\left[\begin{array}{c} n \\ k \end{array}\right]\!\right] \left(\frac{\pi}{R}\right)^{n-k} \mathcal{S}_k^{\vartheta}} \tag{9.5}$$

where[4] $\left[\!\left[\begin{array}{c} n \\ k \end{array}\right]\!\right] \;=\;$ *coefficient of* u^k *in* $\prod_{s=0}^{n-1}(u+s)$. *Moreover,*

$$\boxed{\textit{The expressions reduce to } \mathcal{S}_n^{\vartheta} \textit{ for } R \,=\, \infty}$$

.

Proof. That the expressions give $\mathcal{S}_n^{\sigma}$ for $R = \infty$ follows by construction, but can be verified directly since $\left[\!\left[\begin{array}{c} n \\ n \end{array}\right]\!\right] = 1$. Now, let $x \in \mathbb{R}$ arbitrary and fixed, and consider

$$(T_x \circ g_R)(t) \;=\; x - \frac{R}{\pi}\,\ln\,(1-\frac{\pi\,t}{R}),$$

which is interpreted $x+t$ for $R \,=\, \infty$.

Let h_x be the quotient of the pair of fundamental solutions associated to x as in Proposition 3.1. Then

$$\begin{aligned} \mathcal{S}_n^{\vartheta,\,g_R}(x) \;&\overset{(i)}{=}\; \frac{\partial^n}{\partial\,t^n}\,|_{t=0}\,V\,(x,\,x+g_R(t)) \\ &\overset{(ii)}{=}\; (h_x\circ T_x \circ g_R)^{(n)}\,(0) \\ &=\; \sum_{k\,=\,1}^{n} h_x^{(k)}(x)\;g_{R n,k} \\ &\overset{(iii)}{=}\; \sum_{k\,=\,1}^{n} \mathcal{S}_k^{\vartheta}(x)\;g_{R n,k}\,, \end{aligned} \tag{9.6}$$

where (i) holds by (3.8) and (ii) by (3.13); the coefficients $g_{Rn,k}$ depend on the derivatives of order 1 or higher of g_R at 0 and are computed below; in (iii) we use Proposition 3.1 by which $h_x^{(n)}(x) \;=\; \mathcal{S}_n^{\sigma}(x)$, for all integer $n \;\geq\; 1$.

To find $g_{Rn,k}$, avoiding the Faà di Bruno formula, consider the function of u and v,

$$r_x(u,v) \;=\; e^{\,u\,\frac{\pi}{R}(v-x)}\,,$$

and compute the partial derivative with respect to t of order $n \;\geq\; 1$ at $t=0$ of

$$r_x\,(u, x+\;g_R(t))$$

in two different ways. On the one hand that partial derivative is given in terms of the $g_{Rn,k}$ by

$$\sum_{k\,=\,1}^{n} u^k\,\left(\frac{\pi}{R}\right)^k\,g_{R n,k}\,, \tag{9.7}$$

and on the other hand, since

$$r_x\,(u,\,x+\,g_R(t)) \;=\; \left(1-\frac{\pi\,t}{R}\right)^{-u},$$

that derivative computed directly gives

$$\left(-\frac{\pi}{R}\right)^n\,\prod_{s\,=\,0}^{n-1}(-u-s) \;=\; \left(\frac{\pi}{R}\right)^n\,\prod_{s\,=\,0}^{n-1}(u+s)\,. \tag{9.8}$$

---

[4] $\left[\!\left[\begin{array}{c} n \\ k \end{array}\right]\!\right]$ is the absolute value of what is commonly known as the (n,k)-*Stirling number of the second kind* [1]

Now the coefficient of u^k in (9.7) equals the coefficient of u^k in (9.8) which equals $\left(\frac{\pi}{R}\right)^n$ times the coefficient of u^k in $\prod_{s=0}^{n-1}(u+s)$. Thus

$$g_{R_{n,k}} = \left[\left[\begin{matrix} n \\ k \end{matrix}\right]\right] \left(\frac{\pi}{R}\right)^{n-k} .$$

□

Corollary 9.1.1. *For the choice of g_R given by (9.4) the corresponding g_R-Schwarzian functions in TM are given for* $\mathrm{v} \in TM$

$$\boxed{\mathrm{S}_n^{\sigma,\, g_R}(\mathrm{v}) = \sum_{k=1}^{n} \left[\left[\begin{matrix} n \\ k \end{matrix}\right]\right] \left(\frac{\pi\,\|\mathrm{v}\|}{R}\right)^{n-k} \mathrm{S}_k^{\sigma}(\mathrm{v})} \tag{9.9}$$

where $\mathrm{S}_n^{\sigma}(\mathrm{v})$ *is given for $v \in TM$ by (5.15).*

Proof. Apply Definition 5.3. □

9.2.1. *A few g_R-Schwarzians $\mathcal{S}_n^{\vartheta,\, g_R}$ for $g_R = -\frac{R}{\pi}\ln\left(1-\frac{\pi\, t}{R}\right)$ in terms of ϑ and R.*

$$\mathcal{S}_1^{\vartheta,\, g_R} = 1\,, \quad \mathcal{S}_2^{\vartheta,\, g_R} = \frac{\pi}{R}\,, \quad \mathcal{S}_3^{\vartheta,\, g_R} = \mathcal{S}_3^{\vartheta} + \frac{2\,\pi^2}{R^2} = 2\,\vartheta + \frac{2\,\pi^2}{R^2}\,. \tag{9.10}$$

Note that one recovers (2.2) from

$$\mathcal{D}_2^{\vartheta,\, g_R} = \det \begin{bmatrix} 1 & \frac{\pi}{2R} \\ \frac{\pi}{2R} & \frac{\pi^2}{3\,R^2} + \frac{\vartheta}{3} \end{bmatrix} \geq 0\,. \tag{9.11}$$

In higher order,

$$\begin{aligned} \mathcal{S}_4^{\vartheta,\, g_R} &= \mathcal{S}_4^{\vartheta} + 6\,\mathcal{S}_3^{\vartheta}\,\frac{\pi}{R} + 6\,\frac{\pi^3}{R^3} \\ &= 2\,\vartheta' + 12\,\vartheta\,\frac{\pi}{R} + 6\,\frac{\pi^3}{R^3}, \end{aligned}$$

and

$$\begin{aligned} \mathcal{S}_5^{\vartheta,\, g_R} &= \mathcal{S}_5^{\vartheta} + 10\,\mathcal{S}_4^{\vartheta}\,\frac{\pi}{R} + 35\,\mathcal{S}_3^{\vartheta}\,\frac{\pi^2}{R^2} + 24\,\frac{\pi^4}{R^4} \\ &= 2\,\vartheta'' + 16\,\vartheta^2 + 20\,\vartheta'\,\frac{\pi}{R} + 70\,\vartheta\,\frac{\pi^2}{R^2} + 4\,\frac{\pi^4}{R^4}\,. \end{aligned} \tag{9.12}$$

The corresponding determinantal condition in $\mathcal{D}_3^{\vartheta,\, g_R} \geq 0$ on $\mathbb{R}$, is explicitly computed,

$$\frac{\vartheta^3}{135} - \frac{(\vartheta')^2}{144} + \frac{\vartheta\,\vartheta''}{180} + \frac{(\vartheta'' + 8\,\vartheta^2)\,\pi^2}{720\,R^2} + \frac{\vartheta\,\pi^4}{240\,R^4} + \frac{\pi^6}{2160\,R^6} \geq 0. \tag{9.13}$$

We use this in the next result.

Proposition 9.2. *A condition of second order in the Gauss curvature σ necessary for the existence of the adapted complex structure up to radius R is*[5]

$$32\,\sigma^3 + 6\,\Delta(\sigma^2) + \left(3\,\Delta\sigma + 48\,\sigma^2\right)\frac{\pi^2}{R^2} + 18\,\sigma\,\frac{\pi^4}{R^4} + 2\,\frac{\pi^6}{R^6} \geq 27\,\|\,\mathrm{Grad}\,\sigma\|^2\,, \tag{9.14}$$

Proof. In the expression on $\mathbb{R}$ of $\mathcal{D}_3^{\vartheta,\, g_R}$ given in (9.13) use that

$$\vartheta\,\vartheta'' = \frac{1}{2}\,(\vartheta^2)'' - \vartheta'^2$$

to get

$$2160\,\mathcal{D}_3^{\vartheta,\, R} = 16\,\vartheta^3 - 27\,\vartheta'^2 + 6\,(\vartheta^2)'' + \left(3\,\vartheta'' + 24\,\vartheta^2\right)\frac{\pi^2}{R^2} + 9\,\vartheta\,\frac{\pi^4}{R^4} + \frac{\pi^6}{R^6}\,.$$

[5] The weaker diagonal condition $\mathrm{S}_3^{\sigma,\, g_R} \geq 0$ is also valid and yields

$$\Delta\sigma + 16\,\sigma^2 + 70\,\sigma\,\frac{\pi^2}{R^2} + 24\,\frac{\pi^4}{R^4} \geq 0\,,$$

which in the limit $R = \infty$ reduces to Szőke's inequality (2.3).

The corresponding function in TM restricted to UM is obtained by replacing derivatives of ϑ by covariant derivatives of σ. So, we have, in light of Theorem 2, for all $\mathrm{v} \in UM$, the unit tangent bundle,

$$16\,\sigma^3 + 6\,\nabla^2_{\mathrm{v}}(\sigma^2) + \left(3\,\nabla^2_{\mathrm{v}}\sigma + 24\,\sigma^2\right)\frac{\pi^2}{R^2} + 9\,\sigma\,\frac{\pi^4}{R^4} + \frac{\pi^6}{R^6} \geq 27\,(\nabla_{\mathrm{v}}\sigma)^2 \quad . \tag{9.15}$$

Now, for each $x \in M$, average this inequality over any orthonormal basis $\{\mathrm{v}_1, \mathrm{v}_2\}$ of T_xM; since $\nabla^2_{\mathrm{v}_1}(\sigma^2)+\nabla^2_{\mathrm{v}_2}(\sigma^2) = \Delta(\sigma^2)$ and $(\nabla_{\mathrm{v}_1}\sigma)^2+(\nabla_{\mathrm{v}_2}\sigma)^2 = \|\,\mathrm{Grad}\,\sigma\|^2$ then (9.14) follows. □

Corollary 9.2.1. *If the adapted structure exists up of radius R for M orientable and closed, with* dA *the Riemannian area element, then inequality (2.21) holds.*

Proof. Since if M is closed, by Stokes's Theorem the integrals of $\Delta\sigma$ and $\Delta(\sigma^2)$ vanish. □

Remark 9.1. *It is clear that in (9.14) the equality holds on $\mathbb{R}$ if σ is constant equal to $-\frac{\pi^2}{4\,R^2}$.*

We display two more Schwarzians,

$$\begin{aligned}
\mathcal{S}_6^{\vartheta,\,g_R} &= \mathcal{S}_6^{\vartheta} + 15\,\mathcal{S}_5^{\vartheta}\,\frac{\pi}{R} + 85\,\mathcal{S}_4^{\vartheta}\,\frac{\pi^2}{R^2} + 225\,\mathcal{S}_3^{\vartheta}\,\frac{\pi^3}{R^3} + 120\,\frac{\pi^5}{R^5}\\
&= 2\,\vartheta''' + 52\,\vartheta\,\vartheta' + (30\,\vartheta'' + 240\,\vartheta^2)\,\frac{\pi}{R} + 170\,\vartheta'\,\frac{\pi^2}{R^2} + 450\,\vartheta\,\frac{\pi^3}{R^3} + 120\,\frac{\pi^5}{R^5}\,,
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{S}_7^{\vartheta,\,g_R} &= \mathcal{S}_7^{\vartheta} + 21\,\mathcal{S}_6^{\vartheta}\,\frac{\pi}{R} + 175\,\mathcal{S}_5^{\vartheta}\,\frac{\pi^2}{R^2} + 735\,\mathcal{S}_4^{\vartheta}\,\frac{\pi^3}{R^3} + 1624\,\mathcal{S}_3^{\vartheta}\,\frac{\pi^4}{R^4} + 720\,\frac{\pi^6}{R^6}\,,\\
&= 2\,\vartheta^{(4)} + 76\,\vartheta\,\vartheta^{(2)} + 52\,(\vartheta')^2 + 272\,\vartheta^3 + (42\,\vartheta^{(3)} + 1092\,\vartheta\,\vartheta')\,\frac{\pi}{R}\\
&\qquad + (350\,\vartheta^{(2)} + 2800\,\vartheta^2)\,\frac{\pi^2}{R^2} + 1470\,\vartheta'\,\frac{\pi^3}{R^3} + 3248\,\vartheta\,\frac{\pi^4}{R^4} + 720\,\frac{\pi^6}{R^6}\,.
\end{aligned}$$

9.3. **A formula for $\mathcal{D}_n^{\vartheta,\,g_R}$ circumventing the Schwarzians.**

There is a classical determinantal formula for quadratic forms, that appears in [5], which we will use to find the functions $\mathcal{D}_n^{\vartheta,\,g_R}$ involving directly the solutions of the diffential equation rather than the individual Schwarzians. In fact for real x_1 and x_2 with $|x_1|$ and $|x_2|$ small enough from a computation with power series, that uses $(x_1^n - x_2^n) = (x_1 - x_2)\sum_{k=0}^{n} x_1^{n-k} x_2^k$, and Proposition 3.1,

$$\frac{(h_x\circ T_x\circ g_R)\,(x_1) - (h_x\circ T_x\circ g_R)\,(x_2)}{x_1 - x_2} = \sum_{p,\,q\,=\,0}^{\infty} \mathcal{S}_{i+j+1}^{\vartheta,\,g_R}(x)\,x_1^p\,x_2^q\,, \tag{9.16}$$

and hence we have the following.

Definition 9.1. *For any smooth $f = f(t)$ and integers $n \geq 1$ and $m \geq 1$ let*

$$\mathbf{T}(f,\,n) = \mathbf{T}(f,\,n)\,(t)$$

be the $n \times 2\,n$ matrix whose entries are the functions in the variable t given by

$$\mathbf{T}(f,\,n)_{i,\,j}\,(t) = \begin{cases} 0 \ \ \textit{for } i < j; \\[2ex] \dfrac{f^{(j-i)}(t)}{(j-i)!} \ \ \textit{for } j \geq i\,. \end{cases} \tag{9.17}$$

Proposition 9.3. *Let y_1 and y_2 be any pair of independent solutions of (3.1) (so the constant $W(y_1, y_2) = y_1\,y_2' - y_1'\,y_2 \neq 0$). Then for all integer $n \geq 1$ and all $x \in \mathbb{R}$*

$$\boxed{\mathcal{D}_n^{\vartheta,\,g_R}(x) = \frac{(-1)^{\frac{n(n+1)}{2}}}{W^n(y_1,\,y_2)}\,\det\begin{bmatrix} \mathbf{T}(y_1\circ T_x\circ g_R,\,n)\,(0) \\[1ex] \mathbf{T}(y_2\circ T_x\circ g_R,\,n)\,(0)\end{bmatrix}}\,, \tag{9.18}$$

where the $2n \times 2n$ matrix whose determinant is indicated above is given as a pair of $n \times 2n$ blocks each one as in Definition 9.1.

Proof. On the one hand, if $f = f(t)$ and $g = g(t)$ are smooth, and a, b, c, d any constants,

$$\det \begin{bmatrix} \mathbf{T}(a\,f + b\,g,\, n)(t) \\ \\ \mathbf{T}(c\,f + d\,g,\, n,)(t) \end{bmatrix} = (a\,d - b\,c)^n \, \det \begin{bmatrix} \mathbf{T}(f,\, n)(t) \\ \\ \mathbf{T}(g,\, n)(t) \end{bmatrix}; \tag{9.19}$$

for, by Laplace formula, we can expand $\det \begin{bmatrix} \mathbf{T}(f,\, n)(t) \\ \\ \mathbf{T}(g,\, n)(t) \end{bmatrix}$ as a sum of terms each of which consists of a product of n minors of order 2: one minor formed with elements from rows 1 and n, another one with elements from rows 2 and $n+1$, and so on, up to one minor formed from rows n and $2\,n$; but all these minors are of the form

$$W_{i,j}(f,\, g) \;=\; \frac{1}{i!\,j!}\, \det \begin{bmatrix} f^{(i)} & f^{(j)} \\ g^{(i)} & g^{(j)} \end{bmatrix},$$

and for them it holds that $W_{i,j}(a\,f + b\,g,\, c\,f + d\,g) \;=\; (a\,d - b\,c)\, W_{i,j}(f,\, g)$. On the other hand, if y is any solution of (3.1) and x any point in $\mathbb{R}$,

$$y(t) = y(x)\, f_{x,1}(t) + y'(x)\, f_{x,2}(t)\,.$$

Thus, in light of (9.19), to prove (9.18) it is enough to show that given any $x \in \mathbb{R}$, for all integer $n \,\geq\, 1$, with notation as in Definition 3.3,

$$\det \begin{bmatrix} \mathbf{T}(\, f_{x,1}{\circ} T_x{\circ}\, g_R,\, n)(0) \\ \\ \mathbf{T}(\, f_{x,2}{\circ} T_x{\circ}\, g_R,\, n)(0) \end{bmatrix} \;=\; (-1)^{\frac{n(n+1)}{2}}\, \mathcal{D}_n^{\vartheta,\, g_R}(x). \tag{9.20}$$

But (9.20) is shown by a classical computation with power series and (3.12) as follows.

If $f \;=\; f(t)$ and $g \;=\; g(t)$ are smooth functions, $w \;=\; w(t)$ is given by $f\,w \;=\; g$ and $\equiv 1$ represents the constant function with value 1, then using $g^{(n)}(t) \;=\; \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)}(t)\, w^{(k)}(t)$ one verifies that

$$\begin{bmatrix} \mathbf{T}(f,\, n)(0) \\ \\ \mathbf{T}(g,\, n)(0) \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{T}(\equiv 1,\, n)(0) \\ \\ \mathbf{T}(w,\, n)(0) \end{bmatrix}}_{(*)} \begin{bmatrix} \mathbf{T}(f,\, n)(0) \\ \\ \mathbf{T}(x^n\, f,\, n)(0) \end{bmatrix}. \tag{9.21}$$

For example, for $n \;=\; 2$,

$$\begin{bmatrix} f(0) & f'(0) & \frac{f''(0)}{2!} & \frac{f'''(0)}{3!} \\ 0 & f(0) & f'(0) & \frac{f''(0)}{2!} \\ g(0) & g'(0) & \frac{g''(0)}{2!} & \frac{g'''(0)}{3!} \\ 0 & g(0) & g'(0) & \frac{g''(0)}{2!} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ w(0) & w'(0) & \frac{w''(0)}{2!} & \frac{w'''(0)}{3!} \\ 0 & w(0) & w'(0) & \frac{w''(0)}{2!} \end{bmatrix} \cdot \begin{bmatrix} f(0) & f'(0) & \frac{f''(0)}{2!} & \frac{f'''(0)}{3!} \\ 0 & f(0) & f'(0) & \frac{f''(0)}{2!} \\ 0 & 0 & f(0) & f'(0) \\ 0 & 0 & 0 & f(0) \end{bmatrix}.$$

Now, note that the lower right $n \times n$ block of (*) in (9.21), that is the right $n \times n$ block in

$$\mathbf{T}(w,\, n)(0) \;=\; \left[\begin{array}{ccc|cc} w(0) & & \frac{w^{(n-1)}(0)}{(n-1)!} & \frac{w^{(n)}(0)}{n!} & & \frac{w^{(2\,n-1)}(0)}{(2\,n-1)!} \\ & \ddots & & & \ddots & \\ 0 & & w(0) & w'(0) & & \frac{w^{(n)}(0)}{n!} \end{array}\right],$$

is up to row permutation the $n \times\, n$ the Hankel matrix of $w = w(t)$ at $t \;=\; 0$, the permutation needed being the one that interchanges rows $n-k$ and k of that block for every $k \;=\; 1, \ldots, n-1$ which has a parity with sign $(-1)^{\frac{n(n+1)}{2}}$. Thus, taking determinants in (9.21) with

$$f \;=\; f_{x,1}{\circ} T_x{\circ}\, g_R\,, \qquad g \;=\; f_{x,2}{\circ} T_x{\circ}\, g_R\,, \qquad w \;=\; h_x{\circ} T_x{\circ}\, g_R$$

we see, by 3.12, that the resulting right-hand side agrees with right-handide of equation (9.20), and by construction, so do the left-hand sides. $\square$

9.4. **Schwarzians in $\mathbb{R}$ and in TM for constant curvature.**

When the Gauss curvature σ is constant equalt to K on M, for each integer $n \geq 1$ the function $\mathrm{v} \in TM \to \dfrac{\mathrm{S}_n^{\sigma,\, g_R}(\mathrm{v})}{n!}$ is constant along level sets of $g(\mathrm{v},\mathrm{v})$. Across level sets, due to Proposition 5.3, it varies according to the rule $\lambda \mathrm{v} \in TM \longrightarrow \lambda^{n-1} \dfrac{\mathrm{S}_n^{\sigma,\, g_R}(\mathrm{v})}{n!}$.

To compute these Schwarzians for $R = \infty$ consider $f'' + KF = 0$ and let the function $s, t \mapsto V(x, x+t)$ be as in Definition 3.1 . We have the independent solutions $f_1 = \cos(\sqrt{K} t)$ and $f_2 = \frac{1}{\sqrt{K}} \sin(\sqrt{K} t)$, hence $h(t) = \frac{1}{\sqrt{K}} \tan(\sqrt{K} t)$, interpreted henceforth as $h(t) = t$ for $K = 0$, and $h(t) = \frac{1}{\sqrt{|K|}} \tanh(\sqrt{|K|} t)$ for $K < 0$. Since for any a and b in $\mathbb{C}$, $\tan(a+b) = \dfrac{\tan a \,+\, \tan b}{1 - \tan a \tan b}$, and by (3.5), we have

$$\sum_{n=0}^{\infty} \mathcal{S}_n^K(x) \frac{t^n}{n!} \;=\; V(x,\, x+t) \;=\; \frac{1}{\sqrt{K}} \tan(\sqrt{K} t)\,, \tag{9.22}$$

independent of x as expected. Thus, the Schwarzians in $\mathbb{R}$ are the constants, for integer $n \geq 1$,

$$\begin{aligned} \mathcal{S}_{2n-1}^K(x) \;&=\; \mathcal{S}_{2n-1}^K = \frac{1}{n} \binom{4^n}{2} |B_{2\,n}|\, K^{n-1}\ , \\ \mathcal{S}_{2n}(x) \;&=\; 0\,, \end{aligned} \tag{9.23}$$

with B_n [1] the Bernoulli numbers [6] defined by $\frac{x}{e^x - 1} \;=\; \sum_{n=0}^{\infty} B_n \, \frac{x^n}{n!}$.

According to Definition 5.15, the Schwarzians in TM are obtained via the substitutions in (5.14) which when M has constant curvature $\sigma = K$ reduce to only $K \mapsto \|\mathrm{v}\|^2 K$; thus for all $\mathrm{v} \in TM$ and integer $n \geq 1$,

$$\begin{aligned} \mathrm{S}_{2n-1}^K(\mathrm{v}) \;&=\; \frac{1}{n} \binom{4^n}{2} K^{n-1} |B_{2\,n}|\, \|\mathrm{v}\|^{2n-2}\ , \\ \mathrm{S}_{2n}^K(\mathrm{v}) \;&=\; 0\,. \end{aligned} \tag{9.24}$$

Remark 9.2. *Formulas (9.24) in one direction and $\mathrm{S}_4^{\sigma}(\mathrm{v}) \,=\, 2\,\nabla_{\mathrm{v}}\sigma$ for all $v \in UM$ in the other, show that $2n$-th Schwarzians $\mathrm{S}_{2n}^{\sigma}(\mathrm{v}) \,=\, 0$ for all $\mathrm{v} \in UM$ and all integer $n \geq 1$ if and only if σ is constant.*

Continuing with $\sigma = K$ a constant, and $R = \infty$, so g_R is the identity map, we compute the function

$$\mathrm{D}_n^{\sigma,\, g_R}(\mathrm{v}) \;=\; \mathrm{D}_n^K(\mathrm{v})\,. \tag{9.25}$$

For $K = 0$ the value of (9.25) is 1 for $n = 1$ and 0 for $n \geq 2$.

On the other hand, for $\|\mathrm{v}\| = 1$, the function $\mathrm{D}_n^K(\mathrm{v})$ is the determinant of the Hankel matrix associated to the MacLaurin coefficients of $\frac{1}{\sqrt{K}} \tan \sqrt{K}\, x$ for $K > 0$ and to those of $\frac{1}{\sqrt{|K|}} \tanh(\sqrt{|K|} x)$ for $K < 0$. The values of these determinants are surely known, but unfortunately we were not able to locate this information in the literature. However the following is available to us (Formula (2.1) [15]),

$$\det \left[\frac{1}{(i+j-1)!} \right]_{i,\,j\,=\,1}^{n} \;=\; (-1)^{n(n-1)/2} \prod_{k=1}^{n} \frac{(n-k)!}{(2\,n-k)!}\,. \tag{9.26}$$

We use this formula and Proposition 9.3 to compute our determinants. We get the following.

---

[6]For instance, $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, $B_4 = -1/30$, $B_6 = 1/42$, $B_8 = -1/30$, $B_{10} = 5/66$ and $B_{12} = -691/2730$. So, the first few non-vanishing Schwarzians are

$$\mathcal{S}_1^K = 1,\quad \mathcal{S}_3^K = 2\,K,\quad \mathcal{S}_5^K = 16\,K^2,\quad \mathcal{S}_7^K = 272\,K^3,\quad \mathcal{S}_9^K = 7936\,K^4,\quad \mathcal{S}_{11}^K = 353792\,K^5\,.$$

Proposition 9.4.

$$\boxed{\mathrm{D}_n^K(\mathrm{v}) \;=\; \left(2\,\|\mathrm{v}\|\sqrt{K}\right)^{n(n-1)} \prod_{k=1}^{n} \frac{(n-k)!}{(2\,n-k)!}} \,. \tag{9.27}$$

Proof. Due to (9.22) and our definitions,

$$\begin{aligned} \mathrm{S}_k^K(\mathrm{v}) \;&=\; \frac{1}{\|\mathrm{v}\|\,\sqrt{K}}\,\tan^{(k)}\left(\|\mathrm{v}\|\sqrt{K}\,t\right)|_{t=0} \\ &\overset{(*)}{=}\; 2\,\left(2\,\sqrt{-1}\,\|\mathrm{v}\|\sqrt{K}\right)^{k-1}\,\tanh^{(k)}\left(\frac{t}{2}\right)|_{t=0} \\ &=\; 2\,\left(2\,\sqrt{-1}\,\|\mathrm{v}\|\sqrt{K}\right)^{k-1}\,\left(M\left(e^t-1\right)\right)^{(k)}|_{t=0} \end{aligned} \tag{9.28}$$

where in (*) we perform a linear change of variable involving $\sqrt{-1}$ (hence the chance to hyperbolic tangent) and M is the Moëbius transformation defined by

$$M(z) \;=\; \frac{z}{z+2}\,.$$

Then, in light of Proposition 9.3, we compute

$$\begin{aligned} \mathrm{D}_n^K(\mathrm{v}) \;&=\; \det\left[\frac{\mathrm{S}_{i+i-1}^K(\mathrm{v})}{(i+j-1)!}\right]_{i,\,j=1}^{n} \\ &=\; \frac{2^n\,\left(2\,\sqrt{-1}\,\|\mathrm{v}\|\sqrt{K}\right)^{n\,(n-1)}}{(\det M)^n}\,\det\left[\frac{1}{(i+j-1)!}\right]_{i,\,j=1}^{n}\,. \end{aligned} \tag{9.29}$$

Now (9.26) together with $\det M = 2$ and $\left(\sqrt{-1}\right)^{n(n-1)} \;=\; (-1)^{n(n-1)/2}$ yields (9.27). □

Remark 9.3. *Thus for $K \neq 0$, on $\mathrm{v} \in TM \setminus M$, we have $\mathrm{D}_n^K(\mathrm{v}) \neq 0$ for all integer $n \geq 1$, and hence*

$$\mathrm{Rank}\left[\frac{\mathcal{S}_{i+j-1}^K(\mathrm{v})}{(i+j-1)!}\right]_{i,\,j=1}^{\infty} \;=\; \begin{cases} \infty \text{ for } \mathrm{v} \in TM \setminus M \\ \\ 1 \text{ for } \mathrm{v} \in M \end{cases}\,.$$

Moreover, for $\mathrm{v} \in TM \backslash M$,

$$\mathrm{sign}\left(\mathrm{D}_n^K(\mathrm{v})\right) \;=\; (\mathrm{sign}\,(K))^{\frac{n(n-1)}{2}}\,, \tag{9.30}$$

and so for $K < 0$ the infinite quadratic defined by $\left[\frac{\mathcal{S}_{i+j-1}^K(\mathrm{v})}{(i+j-1)!}\right]_{i,\,j=1}^{\infty}$ is non-degenerate but not positive-definite, preventing the adapted structure to be defined in the entire TM.

Now, let us consider briefly the case R finite. Due to (9.22), taking g_R as in (9.4) the Schwarzians $\mathcal{S}_n^{K,\,g_R}(x)$ at any $x \in \mathbb{R}$ are equal to $\mathcal{S}_n^{K,\,g_R}(0)$, the MacLaurin coefficients in t of

$$-\frac{1}{\sqrt{K}}\tan\left(\frac{\sqrt{K}\,R}{\pi}\,\ln\left(1-\frac{\pi\,t}{R}\right)\right)\,,$$

which can now be written down explicitly from the formulas (9.24) and those in Corollary 9.1.1.

This way we get that the g_R-Schwarzians in TM for constant curvature K on M are given, for $\mathrm{v} \in TM$,

$$\boxed{\mathrm{S}_n^{K,\,g_R}(\mathrm{v}) \;=\; \|\mathrm{v}\|^{n-1} \sum_{l=1}^{\lfloor (n+1)/2 \rfloor} \binom{4^l}{2} \left[\!\left[\begin{matrix} n \\ 2\,l-1 \end{matrix} \right]\!\right] \left(\frac{\pi}{R}\right)^{n-2l+1} \frac{|B_{2\,l}|}{l}\,K^{l-1}}\,, \tag{9.31}$$

where $\lfloor x \rfloor$ indicates the largest integer smaller or equal to x.

For example, for $K=0$ we have, on TM, for all $n \geq 1$

$$\mathrm{S}_n^{K\equiv 0,\, g_R}(\mathrm{v}) \;=\; (n-1)!\,\left(\frac{\pi\,\|\mathrm{v}\|}{R}\right)^{n-1},$$

and hence we compute,

$$\begin{aligned}
\mathrm{D}_n^{K\equiv 0,\, g_R}(\mathrm{v}) \;&=\; \det\left[\frac{\mathrm{S}^{K,\, g_R}_{(i+j-1)}(\mathrm{v})}{(i+j-1)!}\right]^n_{i,\,j=1} \\
&=\; \det\left[\frac{1}{i+j-1}\left(\frac{\pi\,\|\mathrm{v}\|}{R}\right)^{i+j-2}\right]^n_{i,\,j=1} \\
&=\; \left(\frac{\pi\,\|\mathrm{v}\|}{R}\right)^{n(n-1)}\det\left[\frac{1}{i+j-1}\right]^n_{i,\,j=1} \\
&=\; \left(\frac{\pi\,\|\mathrm{v}\|}{R}\right)^{n(n-1)}\frac{[1!\,2!\cdots(n-1)!]^3}{n!\,(n+1)!\cdots(2n-1)!}\,.
\end{aligned} \tag{9.32}$$

Here the value of the last determinant is classical (Part VII, Problem 4 [20]), and the indicated matrix is known as the Hilbert matrix .

10. Schwarzians in TM as moments

There are some straight-forward restatements of Theorem 2 in terms of classical *moment sequences* all based on viewing the g_R-Schwarzians on TM as a family of sequences parametrized by $\mathrm{v} \in TM$ which gives the assignment

$$\mathrm{v} \in TM \to \left\{\frac{\mathrm{S}_n^{\sigma,\, g_R}(\mathrm{v})}{n!}\right\}_{n=1}^{\infty}, \tag{10.1}$$

together with the fact that the determinantal conditions (2.19) characterize *non-negative sequences in* $(-\infty,\infty)$, which are precisely those sequences $\{c_1, c_2, \cdots\}$ expressible as power moments of a non-negative Borel measure [4], [21], [27].

$$c_n \;=\; \int_{-\infty}^{\infty} t^{n-1} d\mu(t)\,.$$

Thus, for each $\mathrm{v} \in TM$ we have a classical moment problem concerning the sequence in (10.1) which is solvable for all v in TM precisely when the adapted complex structure is defined in $T^R M$, and this happens whenever the Schwarzian functions are expressible as the integrals

$$\boxed{\mathrm{S}_n^{\sigma,\, g_R}(\mathrm{v}) \;=\; n!\int_{-1/\rho_{\mathrm{v}}}^{1/\rho_{\mathrm{v}}} t^{n-1}\, d\mu_{\mathrm{v}}^{\sigma,\, g_R}(t)} \tag{10.2}$$

with a non-negative Borel measure $d\mu_{\mathrm{v}}^{\sigma,\, g_R}(t)$ and $\rho_{\mathrm{v}} \;=\; \rho_{\mathrm{v}}^{\sigma,\, g_R} \in (0,\infty]$ both depending on v, and of course the choice of g_R .

Note that the Gauss curvature σ is determined by the integral above with $n=3$ computed point-wise by choosing for each $x \in M$ any non-zero $\mathrm{v} \in T_x M$, and that the measure, at any $\mathrm{v} \in TM$ integrates to the value 1 since $\mathrm{S}_1^{\sigma,\, g_R}(\mathrm{v}) \equiv 1$.

Similarly, the existence of the adapted structure on $T^R M$ is equivalent to the positivity of a classical functional defined by sequences [4].

In fact, introduce the sets of functions

$$\begin{aligned}
\mathcal{P} \;&=\; \{\,\mathrm{p}\colon TM \times \mathbb{R} \to \mathbb{R}\,,\ \text{where}\ \ \mathrm{p}(\mathrm{v},t)\ \text{is polynomial in}\ t\}\,; \\
\mathcal{P}_+ \;&=\; \{\,\mathrm{p} \in \mathcal{P}\,,\ \mathrm{p}(\mathrm{v},t) \geq 0\,,\ \ \text{for all}\ (\mathrm{v},t) \in TM \times \mathbb{R}\}\,.
\end{aligned}$$

Thus p is in $\mathcal{P}$ if and only if

$$\mathrm{p}(\mathrm{v},t) \;=\; \sum_{k=0}^{n} a_k(\mathrm{v})\, t^k$$

for some non-negative integer n where $a_k\colon TM \to \mathrm{R}$ have no particular regularity assumption imposed on them. Also let

$$\begin{aligned}\mathcal{F} &= \{\mathrm{f}\colon TM \to \mathbb{R}\}\,,\\ \mathcal{F}_+ &= \{\mathrm{f} \in \mathcal{F}\,,\ \mathrm{f}(\mathrm{v}) \geq 0\,,\ \mathrm{v} \in TM\}\,,\end{aligned}$$

where no regularity is required here either.

Consider the map

$$\mathfrak{S}^{\sigma,\,g_R}\colon \mathcal{P} \to \mathcal{F} \tag{10.3}$$

obtained by extending $\mathcal{F}$-linearly the assignments

$$\mathfrak{S}^{\sigma,\,g_R}[t^{n-1}](\mathrm{v}) := \frac{\mathrm{S}_n^{\sigma,\,g_R}(\mathrm{v})}{n!}\,,\quad n = 1,\,2,\,\cdots\,. \tag{10.4}$$

Since for all $\mathrm{v} \in TM$ by definition $\mathfrak{S}^{\sigma,\,g_R}[t^0] = \mathrm{S}_1^{\sigma,\,g_R}(\mathrm{v}) \equiv 1$, we have always

$$\mathfrak{S}^{\sigma,\,g_R}\,[\mathcal{P}] = \mathcal{F}\,.$$

On the other hand, the following holds, by Theorem 2.

Corollary 10.0.1. *The adapted complex structure is defined on T^RM if and only if*

$$\mathfrak{S}^{\sigma,\,g_R}\,[\mathcal{P}_+] = \mathcal{F}_+\,.$$

In such case, for any $p \in \mathcal{P}$ and for all $\mathrm{v} \in TM$,

$$\mathfrak{S}^{\sigma,\,g_R}[p](\mathrm{v}) = \int_{-1/\rho_{\mathrm{v}}^{\sigma,\,g_R}}^{1/\rho_{\mathrm{v}}^{\sigma,\,g_R}} p(\mathrm{v},\,t)\;d\mu_{\mathrm{v}}^{\sigma,\,g_R}(t)$$

10.1. Estimate for the zeros of $\mathrm{P}_n^{\sigma,\,g_R}(\mathrm{v},t)$.

One final comment on this transplantation of the moment problem to TM concerns the expressions

$$\mathrm{P}_n^{\sigma,\,g_R}(\mathrm{v},t) := \det\left[\frac{\mathrm{S}_{i+j}^{\sigma,\,g_R}(\mathrm{v})}{(i+j)!} - \frac{\mathrm{S}_{i+j-1}^{\sigma,\,g_R}(\mathrm{v})}{(i+j-1)!}\,t\right]_{i,\,j=1}^{n} \in \mathcal{P}, \tag{10.5}$$

for integer $n \geq 1$. When the support of measure at $\mathrm{v} \in TM$ does not consist of a finite number of points, that is when $\mathrm{R}^{\sigma,\,g_R}(\mathrm{v}) = \infty$ (see 8.1), this infinite set of polynomials constitute along the set $\{\phi_t\mathrm{v},\ \ t \in \mathbb{R}\} \subset TM$ a point-wise orthogonal system with respect to $\mathfrak{S}^{\sigma,\,g_R}$, that is,

$$\mathfrak{S}^{\sigma,\,g_R}\,[\,\mathrm{P}_n^{\sigma,\,g_R}\mathrm{P}_m^{\sigma,\,g_R}\,]\,(\phi_t\mathrm{v}) = \begin{cases}\mathrm{D}_{n-1}^{\sigma,\,g_R}(\phi_t\mathrm{v})\,\mathrm{D}_n^{\sigma,\,g_R}(\phi_t\mathrm{v}) & \text{if } n = m\,,\\[2ex] 0 \quad \text{if } n \neq m\,,\end{cases} \tag{10.6}$$

for all integers $n, m \geq 1$ and all $t \in \mathbb{R}$, or with respect to the measure above

$$(10.6) = \int_{-1/\rho_{\phi_t\mathrm{v}}^{\sigma,\,g_R}}^{1/\rho_{\phi_t\mathrm{v}}^{\sigma,\,g_R}} \mathrm{P}_n^{\sigma,\,g_R}(\phi_t\mathrm{v},\,t)\;\mathrm{P}_m(\phi_t\mathrm{v},\,t)\;d\mu_{\phi_t\mathrm{v}}^{\sigma,\,g_R}(t)\,.$$

Concerning the relation of the support of this measures and the Gauss curvature σ on M note that, by Remark 5.4, the zero section $M \subset TM$ is mapped to the sequence $\{1,\,0,\cdots\}$, and correspondingly, for all $\mathrm{v} \in M \subset TM$,

$$d\mu_{\mathrm{v}}^{\sigma,\,g_R}(t) = \delta(t), \quad \rho_{\mathrm{v}}^{\sigma,\,g_R} = \infty\,, \tag{10.7}$$

Off M, note that by Proposition 5.3, for any λ,

$$\lambda\,\mathrm{v} \in TM \to \{\lambda^{n-1}\,\mathrm{S}_n^{\sigma,\,g_R}(\mathrm{v})/n!\}_{n=1}^{\infty}\,,$$

so, (10.1) and the measure is determined by its restriction to the unit tangent bundle UM. Along this parameter set the sequence and the measure vary, unless the Gauss curvature σ is constant on M.

For $\mathrm{v} \in UM$, and for $g_R(z) = -\dfrac{R}{\pi}\ln\Big(1 - \dfrac{\pi\,z}{R}\Big)$, from the computations on the real line $\mathbb{R}$ given in Proposition 4.1, together with the method in the proof of Theorem 2 by which we

relate those computations to a unit speed geodesic, we readily obtain estimates in terms of the curvature σ of M for the support of $d\mu_{\mathrm{v}}^{\sigma,\, g_R}(t)$, when σ is bounded as in Proposition 4.1.

Now, since for each v the measure $d\mu_{\mathrm{v}}^{\sigma,\, g_R}(t)$ maybe constructed [4] [**?**], [8] by quadratures as a limit of measures defined from the zeros of the sequence $\{\mathrm{P}_n^{\sigma,\, g_R}(\mathrm{v},\, t)\}_{n=1,\ldots}$.

Proposition 10.1 (The zeros of $\mathrm{P}_n^{\sigma,\, g_R}(\mathrm{v}, t)$)**.** *If the adapted structure is defined on $T^R M$, for each $\mathrm{v} \in UM$, the unit tangent bundle, the n zeros of the polynomial in t*

$$\mathrm{P}_n^{\sigma,\, g_R}(\mathrm{v}, t)$$

are simple and real, and all lie in the interval $\left[-\frac{\pi}{R}\,,\ \frac{\pi}{R}\right]$ *as long as* $\sup_{x\in M} \sigma\ \le\ 0$*, or, provided that* $\sup_{x\in M} \sigma\ >\ 0$ *, in the interval* $\left[-\dfrac{\pi}{R - R\, e^{-\frac{\pi^2}{2R\sqrt{\sup \sigma}}}}\,,\ \dfrac{\pi}{R - R\, e^{-\frac{\pi^2}{2R\sqrt{\sup \sigma}}}}\right]$ *which if* $R\ =\ \infty$ *reduces to the interval* $\left[-\dfrac{2\sqrt{\sup \sigma}}{\pi}\,,\ \dfrac{2\sqrt{\sup \sigma}}{\pi}\right],$ *.*

Remark 10.1. *As an illustration, for Gauss curvature $\sigma \equiv 0$ and $g_R\ =\ -\frac{R}{\pi}\ln\left(1 - \frac{\pi z}{R}\right)$ the assignment (10.4) becomes*

$$\mathfrak{S}^{\,\sigma,\, g_R}[t^{n-1}](\mathrm{v})\ =\ \frac{1}{n}\left(\frac{\pi\,\|\mathrm{v}\|}{R}\right)^{n-1}\ ,$$

and thus (10.2) is satisfied with the measure given, for $\mathrm{v} \in TM \setminus M$, by

$$d\mu_{\mathrm{v}}^{K\equiv 0,\, g_R}(t)\ =\ \begin{cases} \dfrac{R}{\pi\,\|\mathrm{v}\|}\, dt \quad \textit{if} \quad 0\ \le\ t\ \le\ \dfrac{\pi\,\|\mathrm{v}\|}{R}\,; \\ \\ 0 \quad \textit{otherwise}\ , \end{cases}$$

and by (10.7) along $M \subset TM$.

Thus, along $\|\mathrm{v}\| = \frac{R}{\pi}$, the polynomials $\mathrm{P}_n^{K\equiv 0,\, g_R}(\mathrm{v}, t)$, orthogonal with respect to the measures above, agree up to constant factors with the shifted Legendre polynomials,

$$P_n(x) = \frac{1}{2^n n!}((u^2-1)^n)^{(n)}|_{u=2x-1}\,.$$

For $\mathrm{v} \neq 0$, the $\mathrm{P}_n^{K\equiv 0,\, g_R}(\mathrm{v}, t)$, have the variables rescaled so that for v along $M \subset TM$ they degenerate to $\mathrm{P}_1^{K\equiv 0,\, g_R}(\mathrm{v}, t)\ =\ t$ and all the other polynomials equal to zero.

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